BELLMAN PARTIAL DIFFERENTIAL EQUATION AND THE HILL PROPERTY FOR CLASSICAL ISOPERIMETRIC PROBLEMS

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Abstract. The goal of this note is to have a systematic approach to generating isoperimetric inequalities from two concrete type of PDEs. We call these PDEs Bellman type because a totally analogous equations happen to rule many sharp estimates for singular integrals in harmonic analysis, and such estimates were obtained with the use of Hamilton–Jacobi–Bellman PDE. We show how classical inequalities of Brascamp–Lieb, Prekopa–Leindler, Ehrhard are particular case of this scheme, which allows us to augment the stock of such inequalities. We approach the isoperimetric inequalities as a maximum (minimum) principle for special types of functions. These functions are compositions of "Bellman function" and an appropriate flow built on test functions. Then the existence of maximum (minimum) principle for such compositions can be reduced to the requirement that Bellman function satisfies a concrete class of nonlinear PDE (written down below). We are left to solve this nonlinear PDE (sometimes a possible task) to enjoy isoperimetric inequalities. The nonlinear PDE that we will describe in this article can be reduced sometimes to solving Laplacian eigenvalue problem, $\bar{\partial}$-equation of certain type or just the linear heat equation.

1. Introduction: what kind of Bellman PDE we consider here.

The papers of Ledoux [22], Barthe [4] and our earlier paper [20] served us as a guide and inspiration for the present article.

In what follows the letter B always stands for a function of n real variables given in some domain $\Omega \subset \mathbb{R}^n$ and satisfying two different but related PDEs. We will describe now these PDEs, they will depend on the choice of matrix $A = (a_1, \ldots, a_n)$ of size $k \times n$, where $k \leq n$, $a_m \in \mathbb{R}^k$ is m-th column vector of A. Both types of PDEs we are interested in here will also depend on a given symmetric real matrix C of size $k \times k$. Practically always this C will be assumed to be positive: $C > 0$ unless we say otherwise, but in fact there are situations where one does not need even nonnegativity, only symmetry would suffice. In this article we assume however $C > 0$, but the reader may consult [20], where one considers arbitrary symmetric C's.

For two matrices M_1, M_2 of the same size $M_1 \bullet M_2$ denotes the Schur product of them, that is entrywise product. Further $a \cdot b$, or $\langle a, b\rangle$ denotes scalar product of vectors in $a, b \in \mathbb{R}^m$.

We will also need several semigroups. If $C \in M_{k\times k}$ and positive, then operator

$$L = L_C := \sum_{i,j=1}^{k} c_{ij} \frac{\partial^2}{\partial x_i \partial x_j}$$

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is a negative generator of the semigroup

$$P_t^C = e^{tL_C}\,.$$

For a nice function $f(x)$ on $\mathbb{R}^k$ the solution of “modified heat equation"

$$\frac{\partial u(x,t)}{\partial t} = (L_C u)(x,t),\ u(x,0) = f(x)$$

will be denoted by $e^{tL_C} f$ and it is quite easy to see how to construct such a solution. Consider $C = I_k$ (identity in $\mathbb{R}^k$) and the usual laplacian $\Delta = L_{I_k}$. Consider the solution $v(x,t)$ of the usual heat equation:

$$\frac{\partial v(x,t)}{\partial t} = (\Delta v)(x,t),\ v(x,0) = f(C^{1/2}x)\,.$$

Then put $u(x,t) = v(C^{-1/2}x,t)$. It solves a modified heat equation. In fact, the Hessian (in variables x) of $v(C^{-1/2}x)$ is $C^{-1/2}(\operatorname{Hess} v)(C^{-1/2}x)C^{-1/2}$, and

$$u(x,t) = \int_{\mathbb{R}^k} f(y) p_t^C(x,y)dy \quad \text{where} \quad p_t^C(x,y) = \frac{\det(C^{-1/2})}{(4\pi t)^{k/2}} e^{-\frac{|C^{-1/2}(x-y)|^2}{4t}}.$$

Therefore,

$$(L_C u)(x,t) = \operatorname{tr}(C \operatorname{Hess} u)(x,t) = \operatorname{tr}(C\, C^{-1/2}(\operatorname{Hess} v)(C^{-1/2}x,t)C^{-1/2}) =$$
$$(\Delta v)(C^{-1/2}x,t) = \frac{\partial v}{\partial t}(C^{-1/2}x,t) = \frac{\partial u}{\partial t}(x,t)\,.$$

1.1. **Special initial data.** We will use very often $P_t^C f$ for f having a special form

$$f(x) := F(a\cdot x)\,,$$

where F is a function of one variable, $x \in \mathbb{R}^k$, and $a \in \mathbb{R}^k$ is a fixed vector. Then the flow $P_t^C f$ can be constructed as follows. Consider $1D$ heat flow (slightly modified):

$$\frac{\partial U}{\partial t}(y,t) = \langle Ca, a\rangle U''(y,t)\,,\ \ y\in\mathbb{R}\,,\ U(y,0) = F(y)\,.$$

Then it turns out that

$$P_t^C f(x) = U(a\cdot x, t) = \int_{\mathbb{R}} F(a\cdot x + y\sqrt{2t\langle Ca,a\rangle})d\gamma_1(y),\ x\in\mathbb{R}^k\,. \tag{1.1}$$

This is of course a simple direct calculation. Notice that if the matrix C is symmetric and $\langle Ca, a\rangle > 0$ but not necessarily nonnegative we can rewrite this as follows

$$\left(\sum_{i,j=1}^k c_{ij}\frac{\partial^2}{\partial x_i \partial x_j} - \frac{\partial}{\partial t}\right) U(a\cdot x,t) = \left(\langle Ca,a\rangle \frac{\partial^2}{\partial y^2} - \frac{\partial}{\partial t}\right) U(a\cdot x,t) = 0\,. \tag{1.2}$$

So in order to construct the flow $P_t^C f(x)$ for this special initial data $f(x) = F(a\cdot x)$ we do not need C to be positive. We only need symmetric C such that $\langle Ca, a\rangle > 0$.

One more nice property of the special initial data is that

$$\nabla P_t^C f(x) = aU'(a\cdot x,t)$$

where $U'(a\cdot x,t) = \frac{\partial}{\partial y}U(y,t)|_{y=a\cdot x}$.

For simplicity we work with *rank-1 case*. Here *rank-1 case* means that we consider initial datas of the form $F(a\cdot x)$, and $a\cdot x$ corresponds to rank-1 linear operator. One can consider *general rank case* i.e., initial data of the form $f(x) = F(xA)$ for some $k\times m$ matrix A and

$x \in \mathbb{R}^k$. For more details we refer the reader to Section 6 where we do consider the *general rank case*.

1.2. **The first type of Bellman PDE: modified concavity property.** Recall that A, C, $C > 0$ are fixed matrices of size $k \times n$ and $k \times k$ correspondingly, and B is a certain (smooth) function given in $\Omega \subset \mathbb{R}^n$. Here is our first PDE, which can be called "modified concavity". We also assume, unless it is said otherwise, that $k \times n$ matrix A has full rank k.

$$\begin{cases} A^*CA \bullet \text{ Hess}\, B(x) \leq 0\,,\ \forall x \in \Omega \\ \det(A^*CA \bullet \text{ Hess}\, B)(x) = 0\,,\ \forall x \in \Omega\,. \end{cases} \tag{1.3}$$

Notice that the first line above is a partial differential inequality (not equation): it is a negative definiteness of a modified Hessian. But we wish to consider the whole system (1.3), for brevity we call it our first Bellman PDE.

We will show below how such B's provide us with important occurrences of isoperimetric inequalities such as Borell's Gaussian noise stability equation, hypercontractivity of Ornstein–Uhlenbeck semigroup, Brascamp–Lieb Gaussian inequalities.

Condition (1.3) (the first inequality) implicitly appears in [11] for the concrete function $B(x_1, x_2, \ldots, x_n) = x_1^{1/p_1} \cdots x_n^{1/p_n}$ and some spacial matrix C. See also [12]. For further details we refer the reader to [20].

But there are instances of very important isoperimetric inequalities, for which the second type of Bellman PDE should be used. It is (1.4) and it gives a bigger amount of Bellman function B's.

1.3. **The second type of Bellman PDE: modified concavity property.** Along with fixed matrices A, C as above, we need the notation D. It is an $n \times n$ non-constant diagonal matrix, where on place $(m, m), m = 1, \ldots, n$, we have $B_m := \frac{\partial B}{\partial x_m}$. We also assume always that $k \times n$ matrix A has full rank k. Here is the second type of Bellman equation, which we assume to hold $\forall x \in \Omega$:

$$\begin{cases} (I - (AD)^*(AD^2A^*)^{-1}AD)(A^*CA \bullet \text{ Hess}\, B)(I - (AD)^*(AD^2A^*)^{-1}AD)(x) \leq 0\,, \\ \det_{n-k}(I - (AD)^*(AD^2A^*)^{-1}AD)(A^*CA \bullet \text{ Hess}\, B)(I - (AD)^*(AD^2A^*)^{-1}AD)(x) = 0\,. \end{cases} \tag{1.4}$$

Here $\det_s$ mean $s \times s$ minors of corresponding matrix-function. Here instead of $C > 0$ we only require $C \geq 0$ and $\langle Ca_j, a_j \rangle > 0$ for all $j = 1, \ldots, n$.

The expression $(I - (AD)^*(AD^2A^*)^{-1}AD)$ seems to be complicated, and also it tacitly assumes the invertibility of $k \times k$ matrix AD^2A^*. In fact, this expression is just precisely the orthogonal projection on the subspace $K(x), x \in \Omega$, where $K(x) = \ker AD(x)$.

Therefore, we can rewrite (1.4) as follows:

$$\begin{cases} P_{K(x)}(A^*CA \bullet \text{ Hess}\, B)P_{K(x)}(x) \leq 0\,,\ \forall x \in \Omega \\ \det_{n-k} P_{K(x)}(A^*CA \bullet \text{ Hess}\, B)P_{K(x)} = 0\,,\ \forall x \in \Omega\,. \end{cases} \tag{1.5}$$

Remark 1. Hence, 1) we should not care too much about the invertibility of AD^2A^*, if it is not invertible, we just understand (1.4) as (1.5); 2) if all entries of $\nabla B(x)$ are non-zero for all $x \in \Omega$ (which is very often the case in applications to isoperimetric inequalities) then AD^2A^* is always invertible (this is just Binet–Cauchy formula and our assumption of full rank of A).

Remark 2. Assume AD has full rank. If $k = n$ then our condition (1.4) becomes trivial and is always true. If $k = n-1$ then P_{ker} has rank 1 and therefore (1.4) holds if and only if

$$\mathrm{Tr}(P_{\text{ker } AD}\,(A^*CA \bullet \mathrm{Hess}\, B)\, P_{\text{ker } AD}) =$$
$$\sum_j B_{jj}\langle Ca_j, a_j\rangle - \sum_{i,j} B_{ij}B_iB_j\langle Ca_i, a_j\rangle\langle (AD^2A^*)^{-1}a_i, a_j\rangle = 0. \tag{1.6}$$

In particular if $B(x_1, \ldots, x_n) = x_n - H(x_1, \ldots, x_{n-1})$ where H is a smooth function of $n-1$ variables such that $\partial_j H \neq 0$ for all $j = 1, \ldots, n-1$, and if $a_1 = (1, 0, \ldots, 0)^T, \ldots, a_{n-1} = (0, \ldots, 0, 1)^T$ (T stands for transposition of rows to columns) is a standard orthonormal basis in $\mathbb{R}^k$, then (1.6) simplifies to

$$\sum_{i,j=1}^{n-1} \frac{\partial_{ij}H}{\partial_i H \partial_j H} a_{ni}a_{nj}c_{ij} = 0. \tag{1.7}$$

This is a direct computations, see the next section for details.

The second type of Bellman PDE (in its form (1.7)) is ruling such isoperimetric inequalities as Prekopa–Leindler inequality (and thus Brunn–Minkowski inequality) and Ehrhard's inequality (see Section 5).

The reader should be warned that even though (1.3) and (1.4) look “almost" the same, they are in fact very different. For example, specially chosen functions B that will prove for us Prekopa–Leindler inequality and Ehrhard's inequality will absolutely not satisfy (1.3) whatever is the choice of $C > 0$, but they will satisfy (1.4) for suitable $C \geq 0$.

Below we start with two examples of using modified concavity Bellman PDE (1.3). Its use will be illustrated by Borell's Gaussian noise stability inequality. We follow closely [22] and [26]. We will also illustrate the use of (1.3) by ultracontractivity property of Ornstein–Uhlenbeck semigroup.

Then we come to PDEs ruling Prekopa–Leindler and Ehrhard's inequalities. These will be of type (1.4).

In Section 7 briefly describes classical isoperimetric inequalities which we have proved in the current paper.

2. Borell's Gaussian noise stability and (1.3) PDE

Here we follow closely the paper of Ledoux [22] and our previous paper [20] in what concerns the use of modified concavity PDE (1.3). We give descriptions in one dimensional case (rank-1 case), and for arbitrary dimension (general rank) we refer the reader to Section 6.

Let X and Y be to standard real Gaussian variable but they are not independent: $\mathbb{E}XY = p, 0 < p < 1$.

One fixes two numbers $u, v \in [0, 1]$ and one looks through all the sets A, B in $\mathbb{R}$ such that

$$\gamma_1(A) = u, \;\; \gamma_1(B) = v,$$

where γ_s is a standard Gaussian measure in $\mathbb{R}^s$. One wishes to solve the following isoperimetric problem: maximize (over A, B) the probability

$$\mathcal{P}(X \in A, Y \in B)\,.$$

First we reformulate the problem in an obvious way, and then we apply (1.3) approach to solve it.

First remark is that we can consider independent standard Gaussians X, Y, but now we look at the pair $X, pX+\sqrt{1-p^2}Y$ and we maximize over $A, B \subset \mathbb{R}^1$

$$\mathcal{P}(X \in A, pX+\sqrt{1-p^2}Y \in B)\,.$$

It is reasonable to think, and this will be proved, that this supremum–let us call it $b(u,v)$–coincides with the following supremum

$$\mathbb{B}^{\sup}(u,v) =$$
$$\sup\left\{\int f(x)g(px+\sqrt{1-p^2}y)d\gamma_2(x,y) : \int f d\gamma_1 = u, \int g d\gamma_1 = v, 0\le f\le 1, 0\le g\le 1\right\}.$$

Of course

$$b(u,v) \le \mathbb{B}^{\sup}(u,v)\,. \tag{2.1}$$

Our goal is to show how using (1.3) we can find the formula for $b(u,v)$ and to prove that $b(u,v) = \mathbb{B}^{\sup}(u,v)$.

Theorem 1. A locally bounded function $B(u_1,\dots,u_n)$ satisfies inequality of (1.3) with matrix A of size $k\times n$ with columns $a_1,\dots a_n$ and $C = I_k$ if and only if

$$\int B(u_1(a_1\cdot x),\dots,u_n(a_n\cdot x))d\gamma_k(x) \le B\left(\int u_1(x)d\gamma_1,\dots,\int u_n(x)d\gamma_1\right),$$

for all smooth bounded functions u_j, where $\|a_i\| = 1, i = 1,\dots,n$.

It is very easy to make a change of variables and to have this result for any $C>0$ and any vectors $a_i \ne 0$:

Corollary 2.1. *Function $B(u_1,\dots,u_n)$ satisfies* (1.3) *(first inequality) with matrix A of size $k\times n$ with columns $a_1,\dots a_n$ and $C>0$ if and only if*

$$\int B(u_1\langle C^{1/2}a_1,x\rangle,\dots,u_n\langle C^{1/2}a_n,x\rangle)d\gamma_k(x) \le$$
$$B\left(\int u_1(x\sqrt{\langle Ca_1,a_1\rangle})d\gamma_1,\dots,\int u_n(x\sqrt{\langle Ca_n,a_n\rangle})d\gamma_1\right).$$

for all smooth bounded functions u_j.

Let us apply Theorem 1 to $\vec{a_1} = (1,0)^T, \vec{a_2} = (p,\sqrt{1-p^2})^T$ and any smooth function $B = B(u,v)$, given on a square $(u,v) \in Q := [0,1]^2$ such that for matrix $A = \begin{bmatrix} 1, & p \\ 0, & \sqrt{1-p^2}\end{bmatrix}$ we have

$$A^*A \bullet \operatorname{Hess} B = \begin{bmatrix} B_{uu}, & pB_{uv} \\ pB_{uv}, & B_{uu}\end{bmatrix} \le 0 \tag{2.2}$$

Then for any such B the theorem claims this inequality :

$$\int B(f(x), g(px+\sqrt{1-p^2}y)\, d\gamma_2(x,y) \le B\left(\int f(x)\, d\gamma_1(x), \int g(x)\, d\gamma_1(x)\right). \tag{2.3}$$

This is the same as

$$\int B(f(x), g(px+\sqrt{1-p^2}y)\, d\gamma_2(x,y) \le B\left(\int f(x)d\gamma_2(x,y), \int g(px+\sqrt{1-p^2}y)d\gamma_2(x,y)\right). \tag{2.4}$$

Let us consider only B on Q, which satisfies (2.2) and also satisfies the following boundary conditions

$$B(0,y)=0, B(1,y)=y, y\in[0,1], B(x,0)=0, B(x,1)=x, x\in[0,1]\,. \tag{2.5}$$

Now we can choose $f=1_E, g=1_F$, where E,F are arbitrary, say, closed sets in $\mathbb{R}^1$.

Then we get from (2.3)

$$\mathcal{P}(X\in E, pX+\sqrt{1-p^2}Y\in F)=\gamma_2(\{(x,y): x\in E, px+\sqrt{1-p^2}y\in F\})\le B(\gamma_1(E),\gamma_1(F))\,, \tag{2.6}$$

or

$$b(u,v)=\sup_{E,F\subset\mathbb{R}^1:\gamma_1(E)=u,\gamma_1(F)=v}\mathcal{P}(X\in E, pX+\sqrt{1-p^2}Y\in F)\le\inf_{B:B\in(2.2),(2.5)}B(u,v)\,. \tag{2.7}$$

Let

$$\Phi(a):=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{a}e^{-x^2/2}dx\,.$$

Let us choose E,F as rays, $E=(-\infty,a), F=(-\infty,b)$, where a,b are chosen $\gamma_1(E)=u, \gamma_1(F)=v$, that is

$$a=\Phi^{-1}(u), b=\Phi^{-1}(v)\,. \tag{2.8}$$

Then

$$\mathcal{P}(X<a, pX+\sqrt{1-p^2}Y<b)\le\inf_{B:B\in(2.2),(2.5)}B(\Phi(a),\Phi(b))\,.$$

We want to show the opposite inequality (thus the equality). It has been made clear above that it is enough to check that the function

$$\mathbb{B}(u,v):=\mathcal{P}(X<\Phi^{-1}(u), pX+\sqrt{1-p^2}Y<\Phi^{-1}(v))$$

satisfies (2.5) and also satisfies (2.2). Relation (2.5) is obvious from the definition of $\mathbb{B}$, we are left to verify (2.2).

Moreover, we will see that $\mathbb{B}$ is "the nest" function satisfying (2.2) and boundary conditions (2.5), in the sense that the following "saturation" of non-positivity of modified Hessian matrix holds:

$$\mathbb{B}_{uu}\mathbb{B}_{vv}-p^2\mathbb{B}_{uv}^2=0\,,\forall(u,v)\in Q\,. \tag{2.9}$$

Remark 3. It is clear that to satisfy inequality (2.2) it is sufficient to satisfy equation (2.9) and inequality $B_{uu}+B_{vv}\le 0$, or even just either $B_{uu}<0$ or $B_{vv}<0$.

To this end we write $\mathbb{B}(u,v)$ in a different form. We change the variable in the integral:

$$\frac{1}{2\pi}\int f(x)g(px+\sqrt{1-p^2}y)e^{-x^2/2}e^{-y^2/2}dxdy=\int f(x)g(z)K_p(x,z)e^{-x^2/2}e^{-z^2/2}dxdz$$

and easily check that

$$K_p(x,z)=\frac{1}{2\pi}e^{-\alpha x^2-\alpha z^2+\frac{2\alpha}{p}xz},\text{ where }\alpha=\frac{p^2}{2(1-p^2)}\,.$$

Plugging into the above formula $f=1_{(-\infty,a)}$, $g=1_{(-\infty,b)}$, $a=\Phi^{-1}(u), b=\Phi^{-1}(v)$, we get

$$2\pi\mathbb{B}(u,v):=2\pi\mathcal{P}(X<\Phi^{-1}(u), pX+\sqrt{1-p^2}Y<\Phi^{-1}(v))= \tag{2.10}$$

$$\int_{-\infty}^{\Phi^{-1}(u)}\int_{-\infty}^{\Phi^{-1}(v)}e^{-\alpha x^2-\alpha z^2-\frac{2\alpha}{p}xz}e^{-x^2/2}e^{-y^2/2}dxdz=\int_{-\infty}^{\Phi^{-1}(u)}\int_{-\infty}^{\Phi^{-1}(v)}K_p(x,z)d\gamma_2(x,z)\,. \tag{2.11}$$

Direct calculation gives (let $\varphi := \Phi'$)

$$\mathbb{B}(u,v) = \mathcal{P}\left(X \le \Phi^{-1}(u),\ Y \le \frac{\Phi^{-1}(v) - pX}{\sqrt{1-p^2}}\right) =$$

$$\int_{-\infty}^{\Phi^{-1}(u)} \int_{-\infty}^{\frac{\Phi^{-1}(v)-ps}{\sqrt{1-p^2}}} \varphi(t)\varphi(s)dtds = \int_{-\infty}^{\Phi^{-1}(v)} \int_{-\infty}^{\frac{\Phi^{-1}(u)-ps}{\sqrt{1-p^2}}} \varphi(t)\varphi(s)dtds;$$

$$\mathbb{B}_u = \int_{-\infty}^{\frac{\Phi^{-1}(v)-p\Phi^{-1}(u)}{\sqrt{1-p^2}}} \varphi(t)dt;$$

$$\mathbb{B}_{uu} = \varphi\left(\frac{\Phi^{-1}(v) - p\Phi^{-1}(u)}{\sqrt{1-p^2}}\right) \frac{-p}{(1-p^2)^{1/2}\varphi(\Phi^{-1}(u))};$$

$$\mathbb{B}_{uv} = \varphi\left(\frac{\Phi^{-1}(v) - p\Phi^{-1}(u)}{\sqrt{1-p^2}}\right) \frac{1}{(1-p^2)^{1/2}\varphi(\Phi^{-1}(v))};$$

$$\mathbb{B}_v = \int_{-\infty}^{\frac{\Phi^{-1}(u)-p\Phi^{-1}(v)}{\sqrt{1-p^2}}} \varphi(t)dt;$$

$$\mathbb{B}_{vv} = \varphi\left(\frac{\Phi^{-1}(u) - p\Phi^{-1}(v)}{\sqrt{1-p^2}}\right) \frac{-p}{(1-p^2)^{1/2}\varphi(\Phi^{-1}(v))};$$

It is clear that $\mathbb{B}_{uu}, \mathbb{B}_{vv} \le 0$ and

$$\mathbb{B}_{uu}\mathbb{B}_{vv} - p^2\mathbb{B}_{uv}^2 = 0.$$

Hence, (2.2) (and also (2.9) are satisfied (so we used the solution of Bellman PDE (1.3) of the first type for our matrix $A = \begin{bmatrix} 1, & \frac{p}{\sqrt{1-p^2}} \\ 0, & \sqrt{1-p^2} \end{bmatrix}$). To prove

$$\inf_{B\in(2.2),(2.5)} B(u,v) = \mathbb{B}(u,v)$$

(that is the first description of $\mathbb{B}$) we used only boundary condition and inequality (2.2). Notice that it is also proved that

$$\mathbb{B}(u,v) = \mathbb{B}^{\sup}(u,v).$$

This is the second description of $\mathbb{B}$.

By Theorem 1 any smooth function B satisfying for all f, g, $0 \le f \le 1, 0 \le g \le 1$,

$$\int B(f(x), g(px + \sqrt{1-p^2}y)\, d\gamma_2(x,y) \le B\left(\int f(x)\, d\gamma_1(x), \int g(x)\, d\gamma_1(x)\right)$$

will also satisfy pointwise inequality (2.2).

This gives the third description of $\mathbb{B}$, it is the saturated (namely, satisfying $\mathbb{B}_{uu}\mathbb{B}_{vv} - p^2\mathbb{B}_{uv}^2 = 0$) solution of (2.2) with boundary condition (2.5). In other words, it is a solution of the first type Bellman equation (1.3) with $A = \begin{bmatrix} 1, & \frac{p}{\sqrt{1-p^2}} \\ 0, & \sqrt{1-p^2} \end{bmatrix}$, which satisfies boundary conditions (2.5).

The fourth description of $\mathbb{B}$ is of course its formula $\mathbb{B}(u,v)=\int_{-\infty}^{\Phi^{-1}(u)}\int_{-\infty}^{\Phi^{-1}(v)}K_p(x,z)d\gamma_2(x,z)=\mathcal{P}(X<\Phi^{-1}(u), pX+\sqrt{1-p^2}Y<\Phi^{-1}(v))$, which we know because this Gaussian extremal problem has been solved beforehand and its solution were known to be rays!

Finally, we can write

$$b=\mathbb{B}^{\sup}=\mathbb{B}=\mathbb{B}^{\inf}\,.$$

Here $\mathbb{B}^{\inf}:=\inf_{B\in(2.2),(2.5)}B(u,v)$. It is interesting to ask how one can find other functions B solving (2.2) and (2.9) simultaneously. We will show how one can do this in Section 2.3.

2.1. **Hypercontractivity of Ornstein–Uhlenbeck semigroup. Young's functions with property (2.9).**

Let us consider again functions B that give us

$$\int_{\mathbb{R}^2}B(\varphi(x),\psi(px+\sqrt{1-p^2}\,y))d\gamma_2\le B\left(\int_{\mathbb{R}^1}\varphi d\gamma_1,\int_{\mathbb{R}^1}\psi d\gamma_1\right).$$

For that we know it is enough to have (2.9) and $B_{uu},B_{vv}\le 0$. Now let us try to choose B in a very simple form

$$B(u,v)=u^{1/a}v^{1/b}\,.$$

It is easy to calulate that (2.9) holds if and only if $1\le a, 1\le b$ and

$$(a-1)(b-1)-p^2\ge 0.$$

This means that if we denote $\varphi=f^a,\psi=g^b$ and choose t from the relationship $p:=e^{-t}$, then we have inequality involving Ornsten–Uhlenbeck semigroup P_t:

$$\int_{\mathbb{R}^n}f\cdot P_tg d\gamma=\int_{\mathbb{R}^{2n}}f(x)g(e^{-t}x+\sqrt{1-e^{-2t}}\,y)d\gamma\le\left(\int_{\mathbb{R}^n}f^ad\gamma\right)^{1/a}\left(\int_{\mathbb{R}^n}g^bd\gamma\right)^{1/b}. \tag{2.12}$$

We obtain hypercontractivity for Ornstein–Uhlenbeck semigroup P_t:

Corollary 2.2.

$$\|P_tg\|_{L^Q(d\gamma)}\le\|g\|_{L^P(d\gamma)},$$

iff

$$Q-1\le e^{2t}(P-1)\,.$$

In fact, taking supremum over $f\in L^a(d\gamma)$ in (2.12) and setting $Q=\frac{a}{a-1}$, $P=b$ we get the inequality of the Corollary under the condition $(a-1)(b-1)-e^{-2t}\ge 0$, which can be rewritten in terms of $P,Q\ge 1$ as $\frac{P-1}{Q-1}-e^{-2t}\ge 0$.

2.2. **The proof of Theorem 1.**

Proof. Let us consider semigroups $P_t:=e^{\Delta_1 t}$, $\mathcal{P}_t:=e^{\Delta_k t}$, where Δ_k is Laplacian in $\mathbb{R}^k$. We already observed the following simple commutation relations: if a,v are vectors in $\mathbb{R}^k$ and $\|a\|=1$ then

$$(P_tF)(a\cdot v)=(\mathcal{P}_tf)(v),\ \text{where}\ f(w):=F(a\cdot w), w\in\mathbb{R}^k\,.$$

The claim of Theorem 1 can be then rewritten as follows

$$(\mathcal{P}_{1/2}B(B(u_1(a_1\cdot w),\dots,u_1(a_1\cdot w)))(0)\le B((\mathcal{P}_{1/2}u_1(a_1\cdot w))(0),\dots(\mathcal{P}_{1/2}u_n(a_n\cdot w))(0)\,,$$

or for shortness just the following inequality with $t=1/2, v=0$:

$$\mathcal{P}_tB(\vec{u})(v)\le B((\mathcal{P}_t\vec{u}))(v)) \tag{2.13}$$

Here the vector function $\vec{u}$ has a special form,

$$\vec{u}(w) := (u_1(a_1 \cdot w), \dots, u_n(a_n \cdot w)), w \in \mathbb{R}^k \,.$$

We will need also (we assume that $u_1, \dots, u_n$ are smooth and bounded)

$$\vec{u'}(w) := (u_1'(a_1 \cdot w), \dots, u_n'(a_n \cdot w)), w \in \mathbb{R}^k \,.$$

Notice that if inequality (2.13) is satisfied for particular t, v, say $t = 1/2$, $v = 0 \in \mathbb{R}^k$, but for all functions $u_1, \dots, u_n$, then it must be automatically satisfied for all $t > 0, v \in \mathbb{R}^k$. Indeed, test the inequality on the shifts and dilations of u_j, namely $\tilde{u}_j(y) = u_j(a_j \cdot v + y\sqrt{2t})$, and use (1.1).

So just as well we need to prove

$$\mathcal{P}_t B(\vec{u})(x) \le B((\mathcal{P}_t \vec{u})(x)) \tag{2.14}$$

for all positive t and all $x \in \mathbb{R}^k$.

To prove (2.13) consider the function in $\mathbb{R}^{k+1}_+$:

$$V(x,t) := B((\mathcal{P}_t \vec{u})(x)) - (\mathcal{P}_t B(\vec{u}))(x), t \ge 0, x \in \mathbb{R}_k$$

and notice that a direct computation gives us the equality

$$\begin{aligned}\big(\Delta_k - \frac{\partial}{\partial t}\big) V(x,t) = \big(\Delta_k - \frac{\partial}{\partial t}\big) B((\mathcal{P}_t \vec{u}))(x)) = \\ \langle A^* A \bullet (\text{Hess} B)(\mathcal{P}_t \vec{u}))(x))(\mathcal{P}_t \vec{u'}))(x), (\mathcal{P}_t \vec{u'}))(x)\rangle \le 0\end{aligned} \tag{2.15}$$

by the first part of our assumption (1.3). Also $V(x, 0) = 0$ obviously. Then by minimum principle (see, for example [25]) we get $V(x,t) \ge 0$ everywhere.

For the converse, we already noticed that inequality in Theorem 1 implies pointwise inequality (2.14), that is $V(x,t) \ge 0$. Now direct computation gives

$$0 \le \lim_{t \to 0} \frac{V(x,t) - V(x,0)}{t} = -\langle A^* A \bullet ((\text{Hess} B)\vec{u}(x))\vec{u'}(x), \vec{u'}(x)\rangle.$$

Since $\vec{u}$ (and hence $\vec{u'}$) is arbitrary Theorem 1 is proved.

□

Remark 4. Suppose C is a $k \times k$ symmetric matrix and $C > 0$. Then we could have consider the semigroup $\mathcal{P}^C_t := e^{tL_C}$, where $L_C := \sum_{i,j=1}^k c_{ij} \frac{\partial^2}{\partial x_i \partial x_j}$. Given that $A^* C A \bullet \text{Hess} B \le 0$ everywhere in Ω we would obtain that the following analog of (2.14) also holds

$$\mathcal{P}^C_t B(\vec{u})(x) \le B((\mathcal{P}^C_t \vec{u})(x)), \ \forall t \ge 0, \forall x \in \mathbb{R}^k \,. \tag{2.16}$$

It is interesting to remark that if we *do not assume* $C > 0$ (or even $C \ge 0$), and we only assume that $\langle C a_j, a_j\rangle > 0$ for all j, then certain shadow of this pointwise inequality still holds. It will be an integral inequality. Notice first that equality

$$\begin{aligned}\big(L_C - \frac{\partial}{\partial t}\big) B(u_1(a_1 \cdot x, t), \dots, u_n(a_n \cdot x, t)) = \\ \langle A^* C A \bullet (\text{ Hess } B)(\vec{u}(x,t))\vec{u'}(x,t), \vec{u'}(x,t)\rangle\end{aligned} \tag{2.17}$$

of course does not require any positivity of C. It follows from (1.2): here each flow $u_j(y,t)$ is with different speed, namely $\langle C a_j, a_j\rangle \frac{\partial^2}{\partial y^2} u_j(y,t) = \frac{\partial}{\partial t} u_j(y,t)$. Then we integrate this equality over $\mathbb{R}^k$. Then

$$\frac{d}{dt} \int_{\mathbb{R}^k} B(u_1(a_1 \cdot x, t), \dots, u_n(a_n \cdot x, t)) dx \ge 0 \,. \tag{2.18}$$

Here one used $\lim_{R\to\infty}\int_{x\in\mathbb{R}^k,:|x|\le R} L_C B(u_1(a_1\cdot x,t),\dots,u_n(a_n\cdot x,t))dx = 0$, which can be seen by Stokes theorem under some mild assumptions on B (see [20]). It is just an integration by parts and the fact that $u(y,t)$ goes to zero fast if y goes to infinity and u is a function with compact support. In particular, if we denote by $\mathbb{E}(t)$ the following "energy" functional

$$\mathbb{E}(t) := \int_{\mathbb{R}^k} B(u_1(a_1\cdot x,t),\dots,u_n(a_n\cdot x,t))dx\,,$$

we obtain the integral inequality

$$\mathbb{E}(t)\ge\mathbb{E}(0)\ \forall t\ge 0\,. \tag{2.19}$$

Of course this inequality immediately follows from the much stronger pointwise inequality (2.16) if $C>0$. To see this just integrate (2.16) with respect to Lebesgue measure dx in $\mathbb{R}^k$.

Remark 5. An interesting (and sometimes useful) observation is that we can think that $P_t,\mathcal{P}_t,\mathcal{P}_t^C$ are semigroups of Ornsein–Uhlenbeck type. We can think that all second order differential operators we used above have a drift (a first order part). Absolutely nothing changes and (2.16) holds. The integral inequalities (2.18), (2.19) will also hold with one small change: the integration should be with respect to the Gaussian measure $d\gamma_k(x)$. Here is a Gaussian analog of (2.19):

$$\mathbb{E}_g(t)\ge\mathbb{E}_g(0)\ \forall t\ge 0\,, \tag{2.20}$$

where

$$\mathbb{E}_g(t) := \int_{\mathbb{R}^k} B((P_t u_1))(a_1\cdot x),\dots,(P_t u_n))(a_n\cdot x))d\gamma_k(x)\,.$$

Of course here P_t is an Ornstein–Uhlenbeck semigroup. Actually it is now a great advantage. We want to make $t\to\infty$ in (2.19) and/or (2.20). It is not so easy to do that in (2.19) (but one can do this sometimes, see [20]), but in (2.20) it is very easy to pass to the limit because measure $d\gamma_k$ is finite and because one has a uniform convergence of $(P_t u)(y)$ to $\int u\,d\gamma_1$ for Ornstein–Uhlenbeck semigroup P_t. Coming to the limit $t\to\infty$ in (2.20) we immediately obtain $E_g(\infty)\ge E_g(0)$ or

$$B\left(\int u_1 d\gamma_1,\dots,\int u_n d\gamma_1\right)\ge\int_{\mathbb{R}^k} B(u_1(a_1\cdot x),\dots,u_n(a_n\cdot x))d\gamma_k(x)\,, \tag{2.21}$$

which gives us another proof of Theorem 1.

2.3. **Solving $B_{uu}B_{vv}-p^2B_{uv}^2=0$.** The following question was asked in [22] and [20].

Problem 1. Describe all possible solutions of the partial differential system of equality and inequality

$$A^*A\bullet\operatorname{Hess}B\le 0\quad\text{and}\quad\det(A^*A\bullet\operatorname{Hess}B)=0.$$

Let us consider the following particular. Let $B\in C^2$ be given in some rectangular domain. Let $n=k=2$ and take $A=(a_1,a_2)$ where $a_1,a_2\neq 0$. Then we must have

$$A^*A\bullet\operatorname{Hess}B=\begin{pmatrix}|a_1|^2B_{11} & a_1\cdot a_2\,B_{12}\\ a_1\cdot a_2\,B_{12} & |a_2|^2B_{22}\end{pmatrix}\le 0\quad\text{and}\quad\det(A^*A\bullet\operatorname{Hess}B)=0.$$

If $a_1\cdot a_2=0$ then B has to be separate concave functions such that $B_{11}B_{22}=0$ and these are the all possible solutions. Therefore we assume that $a_1\cdot a_2\neq 0$. Then we see that B must be separate concave function and moreover

$$\frac{|a_1|^2|a_2|^2}{|a_1\cdot a_2|^2}B_{11}B_{22}-B_{12}^2=0.$$

So in the case $n=k=2$ the problem reduces to the following one

Problem 2. Let $|c| \in [1, \infty)$ and let $B \in C^2$ be given on some rectangular domain in $\mathbb{R}^2$. Characterize all possible separately concave functions B such that

$$c^2 B_{11} B_{22} - B_{12}^2 = 0. \tag{2.22}$$

The case $|c| = 1$ corresponds to the homogeneous Monge–Ampère equation and, thus, to developable surface and the characterization of these surfaces are mostly known. The possible references are Pogorelov [27], Vasyunin–Volberg [28], Ivanisvili et al [15, 16, 17, **?**, 18, 19].

For general $|c| > 1$ we can give local characterization. Namely, we will show that the above equation can be reduced to the following one

$$\frac{\partial f}{\partial \bar{z}} = \bar{f}$$

for some appropriate f (see below).

For separately concave $B(x, y)$ set $B_{xx} = -p^2, B_{yy} = -q^2$. Then equation (2.22) implies that $B_{xy} = cpq$. We also have

$$-2pp_y = cqp_x + cpq_x, \tag{2.23}$$

$$-2qq_x = cqp_y + cpq_y. \tag{2.24}$$

Further we assume that $p, q \neq 0$. Assume that locally the map $p, q : (x, y) \to \mathbb{R}^2$ is invertible, and let (x, y) be its inverse map. Then

$$\begin{pmatrix} p_x & p_y \\ q_x & q_y \end{pmatrix} = \begin{pmatrix} x_p & x_q \\ y_p & y_q \end{pmatrix}^{-1} = \frac{1}{\det(\mathrm{Jacob}(x,y))} \cdot \begin{pmatrix} y_q & -x_q \\ -y_p & x_p \end{pmatrix}.$$

Therefore equations (2.23) and (2.24) take the following form

$$2px_q = cqy_q - cpy_p,$$
$$2qy_p = -cqx_q + cpx_p.$$

This can be written as follows

$$2(px)_q = c(qy)_q - c(py)_p,$$
$$2(qy)_p = -c(qx)_q + c(px)_p.$$

We set $\tilde{U}(p, q) = px(p, q)$ and $\tilde{V}(p, q) = qy(p, q)$. Then we obtain

$$2\tilde{U}_q = c\tilde{V}_q - c\left(\frac{\tilde{V}p}{q}\right)_p,$$

$$2\tilde{V}_p = -c\left(\frac{\tilde{U}q}{p}\right)_q + c\tilde{U}_p.$$

After the logarithmic substitution $\tilde{U}(p, q) = M(\ln p, \ln q)$ and $\tilde{V}(p, q) = N(\ln p, \ln q)$ we obtain the linear equation

$$2M_2 = c(N_2 - N - N_1),$$
$$2N_1 = c(-M - M_2 + M_1).$$

By setting $k = 2/c \in (-2, 2)$, this can be rewritten as follows

$$\begin{pmatrix} N \\ M \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ -k & 0 \end{pmatrix} \begin{pmatrix} N_1 \\ N_2 \end{pmatrix} + \begin{pmatrix} 0 & -k \\ 1 & -1 \end{pmatrix} \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}.$$

We need the following technical lemma.

Lemma 1. If the vector function $\vec{N}(x, y) = (N, M) : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$ satisfies the following first order system of linear differential equations

$$\begin{pmatrix} N \\ M \end{pmatrix} = P \begin{pmatrix} N_1 \\ N_2 \end{pmatrix} + Q \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}.$$

for some invertible 2×2 matrices P, Q where

$$QP^{-1} = \begin{pmatrix} -2t & \delta^2 \\ -1 & 0 \end{pmatrix} \tag{2.25}$$

for some $t \in (-\delta, \delta), \delta > 0$ then after making change of variables $\vec{N}(\vec{x}) = B\vec{U}(A\vec{x})$, where

$$\vec{U} = (U, V)\,,$$

$$B = \begin{pmatrix} t & \sqrt{\delta^2 - t^2} \\ 1 & 0 \end{pmatrix} \quad \text{and}\,,$$

$$A^T = \frac{1}{2} P^{-1} \begin{pmatrix} -1 & -\frac{t}{\sqrt{\delta^2 - t^2}} \\ 0 & -\frac{1}{\sqrt{\delta^2 - t^2}} \end{pmatrix}, \tag{2.26}$$

we obtain

$$\frac{\partial f}{\partial \bar{z}} = \bar{f},$$

where $f = U + iV$.

Proof. Set $P = (P_1, P_2), Q = (Q_1, Q_2)$ where P_i, Q_j are columns.

$$\begin{pmatrix} N \\ M \end{pmatrix} = N_1 P_1 + N_2 P_2 + M_1 Q_1 + M_2 Q_2.$$

Now let $N(x, y) = \tilde{N}(\alpha_1 x + \alpha_2 y, \beta_1 x + \beta_2 y)$ then

$$\begin{aligned} N_1 &= \alpha_1 \tilde{N}_1 + \beta_1 \tilde{N}_2; \\ N_2 &= \alpha_2 \tilde{N}_1 + \beta_2 \tilde{N}_2; \\ M_1 &= \alpha_1 \tilde{M}_1 + \beta_1 \tilde{M}_2; \\ M_2 &= \alpha_2 \tilde{M}_1 + \beta_2 \tilde{M}_2. \end{aligned}$$

So we obtain

$$\begin{aligned} \begin{pmatrix} \tilde{N} \\ \tilde{M} \end{pmatrix} =& (P_1 \alpha_1 + P_2 \alpha_2) \tilde{N}_1 + (P_1 \beta_1 + P_2 \beta_2) \tilde{N}_2 + \\ & (Q_1 \alpha_1 + Q_2 \alpha_2) \tilde{M}_1 + (Q_1 \beta_1 + Q_2 \beta_2) \tilde{M}_2. \end{aligned}$$

Finally we set

$$\tilde{N} = a_1 U + b_1 V$$
$$\tilde{M} = a_2 U + b_2 V.$$

and

$$B = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}.$$

Thus we obtain

$$\begin{aligned}
&\begin{pmatrix} U \\ V \end{pmatrix} = \\
&B^{-1}[a_1(P_1\alpha_1 + P_2\alpha_2) + a_2(Q_1\alpha_1 + Q_2\alpha_2)]U_1 + \\
&B^{-1}[a_1(P_1\beta_1 + P_2\beta_2) + a_2(Q_1\beta_1 + Q_2\beta_2)]U_2 + \\
&B^{-1}[b_1(P_1\alpha_1 + P_2\alpha_2) + b_2(Q_1\alpha_1 + Q_2\alpha_2)]V_1 + \\
&B^{-1}[b_1(P_1\beta_1 + P_2\beta_2) + b_2(Q_1\beta_1 + Q_2\beta_2)]V_2.
\end{aligned}$$

And we would like to see that

$$\begin{pmatrix} U \\ V \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} U_1 \\ U_2 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} V_1 \\ V_2 \end{pmatrix}.$$

This can hold if and only if

$$(P\alpha, Q\alpha) = \frac{1}{2}B\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}B^{-1} = \frac{1}{2}BI^+B^{-1};$$
$$(P\beta, Q\beta) = \frac{1}{2}B\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}B^{-1} = \frac{1}{2}BI^-B^{-1},$$

where

$$\alpha = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}; \quad \beta = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}.$$

Let $e_1 = (1, 0), e_2 = (0, 1)$, and let us introduce the matrices

$$B_1 = \frac{1}{2}\left(BI^+B^{-1}e_1, BI^-B^{-1}e_1\right),$$
$$B_2 = \frac{1}{2}\left(BI^+B^{-1}e_2, BI^-B^{-1}e_2\right).$$

Then the above conditions hold iff

$$P \cdot (\alpha, \beta) = (P\alpha, P\beta) = B_1; \; Q \cdot (\alpha, \beta) = (Q\alpha, Q\beta) = B_2. \tag{2.27}$$

The system (2.27) is overdetermined, it has two equations on one matrix (α, β). There is one compatibility condition: $QP^{-1} = B_2 B_1^{-1}$.

It is easy to calculate B_1, B_2 for matrix B, which was given in the assumption of the lemma. Then we can calculate $B_2 B_1^{-1}$ and automatically obtain that it is equal to

$$\begin{pmatrix} -2t & \delta^2 \\ -1 & 0 \end{pmatrix}.$$

(Note that if r and s are corresponding rows of the matrix B then

$$B_2 B_1^{-1} = \begin{pmatrix} -2\frac{r\cdot s}{|s|^2} & \frac{|r|^2}{|s|^2} \\ -1 & 0 \end{pmatrix},$$

so the claim follows.) This is precisely the form of QP^{-1} from (2.25). This means that the system of equations on matrix (α,β) is compatible, and so matrix (α,β) is well defined.

Set $(\alpha,\beta) := P^{-1}B_1$. This is precisely the formula (2.26). The lemma is proved. $\square$

In our case of P, Q's, we have $P = \begin{pmatrix} -1 & 1 \\ -k & 0 \end{pmatrix}$, $Q = \begin{pmatrix} 0 & -k \\ 1 & -1 \end{pmatrix}$, and

$$QP^{-1} = \begin{pmatrix} -k & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -\frac{2}{c} & 1 \\ -1 & 0 \end{pmatrix},$$

therefore we can apply the lemma and we see that taking $t = 1/c \in (-1,1)$ and $\delta = 1$ we have

$$B = \begin{pmatrix} t & \sqrt{1-t^2} \\ 1 & 0 \end{pmatrix} \quad A = \begin{pmatrix} 0 & -\frac{1}{2} \\ \frac{1}{4t\sqrt{1-t^2}} & -\frac{1}{4}\frac{2t^2-1}{t\sqrt{1-t^2}} \end{pmatrix}.$$

This means that if we set

$$\begin{pmatrix} N(x,y) \\ M(x,y) \end{pmatrix} = \begin{pmatrix} t & \sqrt{1-t^2} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} U\left(-\frac{y}{2}, \frac{x-y(2t^2-1)}{4t\sqrt{1-t^2}}\right) \\ V\left(-\frac{y}{2}, \frac{x-y(2t^2-1)}{4t\sqrt{1-t^2}}\right) \end{pmatrix},$$

where by setting $z = x+iy$ for the function $f(z,\bar{z}) = U(x,y) + iV(x,y)$ we have

$$\frac{\partial f}{\partial \bar{z}} = \bar{f}.$$

It is known that all C^1 solutions of the above equation are real analytic and they can be represented in terms of power series

$$f(z) = \sum_{k=0}^{\infty} c_k J^{(k)}(z\bar{z}) z^k + \bar{c}_k J^{(k+1)}(z\bar{z})\bar{z}^{k+1},$$

where $J(r)$ is modified Bessel I-functions whose series representation is

$$J(r) = \sum_{j=0}^{\infty} \frac{r^j}{(j!)^2}.$$

3. Bellman equation of the second type: PDE that rules the Prekopa–Leindler inequality and Ehrhard inequality

In the previous section we used the following minimum principle. If a smooth function $V(x,t), x \in \mathbb{R}^k, t \geq 0$, satisfies the growth condition $V(x,t) \geq -Me^{\lambda|x|^2}$ for some nonnegative constants $M, \lambda \geq 0$, and it is a superharmonic function in this sense

$$(L_C - \frac{\partial}{\partial t})V(x,t) = \left(\sum_{i,j=1}^{k} c_{ij}\frac{\partial^2}{\partial x_i \partial x_j} - \frac{\partial}{\partial t}\right) V(x,t) \leq 0,\ \forall x \in \mathbb{R}^k, t > 0 \tag{3.1}$$

$C > 0$ then it has minimum principle: $V(x,t) \geq \inf V(\cdot, 0)$.

Remember that our V was of the following type

$$V(x,t) := B((\mathcal{P}_t^C u_1(a_1\cdot .))(x), \dots, (\mathcal{P}_t^C u_n(a_n\cdot .))(x)) - \mathcal{P}_t(B(\vec{u}))(x)\,.$$

The requirement (3.1) transforms into $A^*CA \bullet \mathrm{Hess}B \le 0$. For given A there could be very limited amount of functions B for which there exists a positive C with this property. In fact, in [20] we proved that sometimes one can enumerate all such functions by the list of Young's functions. These functions provide us with Brascamp–Lieb inequality inequality (see [2, 3, 8, 9, 6, 7])

However, to have the minimum principle one does not need V to satisfy (3.1) for all x, t. It is easy to see that it is sufficient to satisfy (3.1) *only* at the points, where $V(x,t)$ ($t > 0$ is fixed) has a local minimum in x. In particular, it is enough to have $C \ge 0$ such that for any (x_0, t_0), $t_0 > 0$,

$$\nabla_x V(x_0,t_0) = 0,\ \mathrm{Hess}_x V(x_0,t_0) \ge 0 \Rightarrow (L_C - \frac{\partial}{\partial t})V(x_0,t_0) \le 0\,. \tag{3.2}$$

This property (we call it *hill property*) was used by Barthe (see [4]) to give a proof of Ehrhard's inequality. Here we will show how the hill property proves such classical inequalities as Prekopa–Leindler and Ehrhard's inequalities and also gives a whole plethora of isoperimetric inequalities ruled by certain PDE. This PDE will be (1.4).

The reason why the hill property works so well is that checking it allows to have a much bigger supply of functions B such that

$$V(x,t) := B((\mathcal{P}_t^C u_1(a_1\cdot .))(x), \dots, (\mathcal{P}_t^C u_n(a_n\cdot .))(x))$$

satisfies the hill property. It turns out that there exists a simple and often easily checkable for concrete functions B PDE (1.4), which is *sufficient* for (3.2) if

$$V(x,t) := B((\mathcal{P}_t^C u_1(a_1\cdot .))(x), \dots, (\mathcal{P}_t^C u_n(a_n\cdot .))(x))\,.$$

Let us prove this statement

In what follows we will need the following conditions at infinity:

$$\forall T > 0,\ \liminf_{|x|\to\infty} \inf_{t\in[0,T]} V(x,t) \ge 0\,. \tag{3.3}$$

Theorem 2. The following statements hold:

(i) If $V(x,t)$ has the hill property (3.2) with certain $C \ge 0$ and also property (3.3) at infinity then $V(x,0) \ge 0$ for all x implies $V(x,t) \ge 0$ for all x.

(ii) Let $C \ge 0$ be $k \times k$ matrix such that $\langle Ca_j, a_j\rangle > 0$ for all j, and the first line of (1.4) is satisfied for B. Then $V(x,t) := B((\mathcal{P}_t^C u_1(a_1\cdot .))(x), \dots, (\mathcal{P}_t^C u_n(a_n\cdot .))(x))$ has the hill property. Moreover, if in addition V has property (3.3) at infinity then $V(x,0) \ge 0$ for all x implies $V(x,t) \ge 0$ for all x. In particular, for $x = 0$ and $t = 1/2$ we have

$$B\left(\int_{\mathbb{R}} u_1(y\sqrt{\langle Ca_1, a_1\rangle})d\gamma_1(y), \dots, \int_{\mathbb{R}} u_1(y\sqrt{\langle Ca_n, a_n\rangle})d\gamma_1(y)\right) \ge 0. \tag{3.4}$$

Proof.

(i) We check that the condition at infinity and the hill property imply the minimum principle: $V(x,t) \ge 0$ for all x. Here we follow the proof of Barthe [3]. It is enough to show that for any $\varepsilon > 0$ we have $V_\varepsilon(x,t) := V(x,t) + \varepsilon t \ge 0$.

First we check that for any T, $V_\varepsilon(x,t)$ does not attain local minimum in $\mathbb{R}^k \times (0,T]$. Indeed, if it does attain a local minimum at point (x_0,t_0) then $\mathrm{Hess}_x V_\varepsilon(x_0,t_0) = \mathrm{Hess}_x V(x_0,t_0) \ge$

$0, \nabla_x V_\varepsilon(x_0,t_0) = \nabla_x V(x_0,t_0) = 0$ and $\partial_t V_\varepsilon(x_0,t_0) = \partial_t V(x_0,t_0)+\varepsilon = 0$. The last property implies that $\partial_t V(x_0,t_0) = -\varepsilon < 0$. However, the hill property implies that $(L_C - \partial_t)V(x_0,t_0) \le 0$. But notice that $L_C V(x_0,t_0) = \mathrm{tr}(C\,\mathrm{Hess}_x V)(x_0,t_0) \ge 0$ since $C \ge 0$ and $\mathrm{Hess}_x V(x_0,t_0) \ge 0$. Putting things together we obtain $\partial_t V(x_0,t_0)\ge 0$, and this contradicts to the fact that $\partial_t V(x_0,t_0) = -\varepsilon$.

Now suppose $V(x_1,t_1) = -\delta, \delta>0$. Then for very small ε, $V_\varepsilon(x_1,t_1)<0$. Taking into account that $V(x,0)\ge 0$ and assumption (3.3) we conclude that V_ε must have a local minimum in $\mathbb{R}^k \times (0,t_1]$. This is a contradiction.

(ii) Let (1.4) be satisfied (just its first line). Let D denotes $n\times n$ diagonal matrix such that it has ∇B on the diagonal, and let I denotes identity matrix. Let

$$V(x,t) := B((\mathcal{P}_t^C u_1(a_1\cdot .))(x), \ldots, (\mathcal{P}_t^C u_n(a_n\cdot .))(x))\,.$$

Let us rewrite $V(x,t)$ in the following form.

$$V(x,t) = B(u_1(a_1\cdot x,t),\ldots,u_n(a_n\cdot x,t))\,, \tag{3.5}$$

where $u_j(z,t)$ denotes $\frac{\partial}{\partial t}u_j(y,t) = \langle Ca_j,a_j\rangle u_j''(y,t)$.

Note that $\nabla_x V = 0$ implies that $\sum_p \frac{\partial B}{\partial u_p}u_p' a_{pj} = 0$ for all $j=1,\ldots,k$. This means that $ADu'=0$ where $u'=(u_1',\ldots,u_n')$. Condition $ADu'=0$ implies that $u' = P_{\ker AD}\,\mathbf{y}$ for some $\mathbf{y}\in\mathbb{R}^n$. After this the direct computations show

$$Tr(C\ \mathrm{Hess}_x\ V(x,t)) - \frac{\partial V}{\partial t} = \left(\sum_{i,j} c_{ij}\frac{\partial^2}{\partial x_i \partial x_j} - \frac{\partial}{\partial t}\right) V = \langle A^* C A \bullet \mathrm{Hess}\, B\, u', u'\rangle =$$
$$\langle P_{\ker AD}\,(A^*CA\bullet \mathrm{Hess}\,B)\,P_{\ker AD}\,\mathbf{y},\mathbf{y}\rangle \le 0.$$

The second line is precisely the first part of (1.4), which we assumed in the theorem.

□

For $V(x,t) := B(u_1(a_1\cdot x,t),\ldots,u_n(a_n\cdot x,t))$, let us see that (3.3) is practically implied by the requirement that $V(x,0)\ge 0$ for all $x\in\mathbb{R}^k$.

We would like to show that for any $T>0$ we have $\liminf_{|x|\to\infty}\inf_{0\le t\le T} B((P_t^C \vec{u})(x)) \ge 0$:

$$\liminf_{|x|\to\infty}\inf_{0\le t\le T} B\left(\ldots, \frac{1}{\sqrt{4\pi\langle Ca_j,a_j\rangle t}}\int_{\mathbb{R}} u_j(y) e^{-\frac{(\langle a_j,x\rangle - y)^2}{4t\langle Ca_j,a_j\rangle}}dy,\ldots\right) \ge 0.$$

For a bounded function u_j with compact support

$$\lim_{|a_j\cdot x|\to\infty}\sup_{t\in[0,T]}\Big|\frac{1}{\sqrt{4\pi\langle Ca_j,a_j\rangle t}}\int_{\mathbb{R}} u_j(y)e^{-\frac{(\langle a_j,x\rangle-y)^2}{4t\langle Ca_j,a_j\rangle}}dy\Big| = 0.$$

Then $0\le V(x,0) = B((u_1)(a_1\cdot x),\ldots,(u_n)(a_n\cdot x)) \to B(0)$ if $x\to\infty$ in such a way that $\min_{i\in[1,\ldots,k]}|a_i\cdot x|\to\infty$. Then

$$\lim_{\min_{i\in[1,\ldots,k]}|a_i\cdot x|\to\infty} V(x,t) = \lim_{\min_{i\in[1,\ldots,k]}|a_i\cdot x|\to\infty} B((P_t^C u_1)(a_1\cdot x),\ldots,(P_t^C u_n)(a_n\cdot x)) = B(0)\ge 0\,.$$

However, we need (3.3), which is a stronger property. Let us assume to this end that on its domain of definition (usually a bounded subset of $\mathbb{R}^n$) function B satisfies

$$\liminf_{|x|\to\infty} \inf_{u_j} B\left(\frac{u_1}{1+|a_1\cdot x|},\dots,\frac{u_n}{1+|a_n\cdot x|}\right)\geq 0\,. \tag{3.6}$$

Here $\inf_{u_j}$ is taken over the ranges of the functions $u_j(\cdot)$ i.e., $\inf_{u_j}=\inf_{u_1,\dots,u_n\,:\,u_j\in\mathrm{range}(u_j(\cdot))}$. Notice that this is a property of B and vectors a_i and not just B alone.

Example 1. We will be using (see Section 5) such B:

$$B(u_1,u_2,u_3):=u_3-\Phi(\alpha_1\Phi^{-1}(u_1)+\alpha_2\Phi^{-1}(u_2))\,,$$

(where $\Phi(x):=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^x e^{-s^2/2}ds$) with some constants $\alpha_1>0,\alpha_2>0$. The domain of definition will be cube $Q=[0,1-\delta]^3$ for any $0<\delta<1$. Let $a_1=(1,0)^T$, $a_2=(0,1)^T$ and $a_3=(a_{31},a_{32})^T$ where $a_{31},a_{32}\neq 0$. Since we cannot find a vector $x\in\mathbb{R}^2$ which will be simultaneously orthogonal to a_1,a_3 (or a_2,a_3) then (3.6) is satisfied as

$$\varepsilon-\Phi(\alpha_1\Phi^{-1}(\varepsilon)+\alpha_2\Phi^{-1}(u_2))\approx-\Phi(-\infty+\alpha_2\Phi^{-1}(u_2))=-\Phi(-\infty)=0\,,$$

and symmetric claim holds for u_2. So the assumption at infinity (3.3) will follow.

4. Simplifications and reductions of the Bellman equation of the second type: (1.4)

Let D denote the diagonal matrix-function with $\nabla B=(B_1,\dots,B_n)$ on the diagonal. We usually assume that A has a full rank. Now we assume that AD has full rank in the domain of definition of B. In applications this routinely happens. If $k=n$ then our condition becomes trivial and is always true. If $k=n-1$ then $P_{\ker}$ has rank 1 and therefore the first line of (1.4) holds if and only if

$$\begin{aligned}&\mathrm{Tr}(P_{\ker AD}\,(A^*CA\bullet\mathrm{Hess}\,B)\,P_{\ker AD})=\\&\sum_j B_{jj}\langle Ca_j,a_j\rangle-\sum_{i,j}B_{ij}B_iB_j\langle Ca_i,a_j\rangle\langle(AD^2A^*)^{-1}a_i,a_j\rangle\leq 0.\end{aligned}$$

If $k=1$ then $d(A)\,\mathrm{Hess}\,B\;d(A)\leq 0$ on the orthogonal complement of the 1-dimensional space $(a_1B_1,\dots,a_nB_n)$, where $d(A)$ is diagonal matrix having on the diagonal a_j. For example, if $a_j=1$ for all j this means that $\mathrm{Hess}\,B\leq 0$ on the variable subspace orthogonal to ∇B.

4.1. **Applications to special B's when $k=n-1$.**

Assume $k=n-1$. Assume also $a_j=e_j$ for $j=1,\dots,n-1$, where e_j are basis vectors. Since $(n-1)\times(n-1)$ matrix AD^2A^* has the following form

$$AD^2A^*=B_n^2a_na_n^T+D_2^2$$

where D_2 is diagonal matrices consisting of elements $B_1,\dots,B_{n-1}$ on the diagonal, we obtain by Sherman–Morison formula

$$(B_n^2a_na_n^T+D_2^2)^{-1}=D_2^{-2}-\frac{B_n^2D_2^{-2}a_na_n^TD_2^{-2}}{1+B_n^2a_n^TD_2^{-2}a_n}, \tag{4.1}$$

therefore

$$\mathrm{Tr}(P_{\ker AD}\,(A^*CA \bullet \mathrm{Hess}\, B)\, P_{\ker AD}) = \sum \langle Ca_j, a_j\rangle (B_{jj} - B_{jj}B_j^2 a_j^T D_2^{-2} a_j) +$$

$$\sum \langle Ca_j, a_j\rangle \frac{B_{jj}B_j^2 B_n^2 |a_n^T D_2^{-2} a_j|^2}{1 + B_n^2 a_n^T D_2^{-2} a_n} - \sum_{i\neq j} B_{ij} B_i B_j \langle Ca_i, a_j\rangle \langle (AD^2A^*)^{-1} a_i, a_j\rangle =$$

$$\frac{1}{B_n^{-2} + a_n^T D_2^{-2} a_n}\left[\langle Ca_n, a_n\rangle \frac{B_{nn}}{B_n^2} + \sum_{j=1}^{n-1} \langle Ca_j, a_j\rangle \frac{B_{jj}}{B_j^2} a_{nj}^2\right] - \sum_{i\neq j} B_{ij} B_i B_j \langle Ca_i, a_j\rangle \langle (AD^2A^*)^{-1} a_i, a_j\rangle$$

Notice that if $B(u_1, \ldots, u_n) = u_n - H(u_1, \ldots, u_{n-1})$ then using (4.1) again we get

$$\mathrm{Tr}(P_{\ker AD}\,(A^*CA \bullet \mathrm{Hess}\, B)\, P_{\ker AD}) = \frac{-1}{1 + a_n^T D_2^{-2} a_n}\left(\sum_{i,j=1}^{n-1} \frac{H_{ij}}{H_i H_j} a_{ni} a_{nj} c_{ij}\right). \tag{4.2}$$

In fact, $B_{nj} = B_{in} = 0$, and (4.1) gives us for $i \neq j, i \neq n, j \neq n$,

$$\langle (AD^2A^*)^{-1} a_i, a_j\rangle = \left\langle (B_n^2 a_n a_n^T + D_2^2)^{-1} a_i, a_j \right\rangle =$$

$$-\left\langle \frac{B_n^2 D_2^{-2} a_n a_n^T D_2^{-2}}{1 + B_n^2 a_n^T D_2^{-2} a_n} a_i, a_j \right\rangle = \frac{\langle D_2^{-2} a_n a_n^T D_2^{-2} a_i, a_j\rangle}{B_n^{-2} + a_n^T D_2^{-2} a_n} = \frac{a_{ni} a_{nj}}{B_i^2 B_j^2 (B_n^{-2} + a_n^T D_2^{-2} a_n)},$$

because $a_i = e_i, 1 \leq i \leq n-1$, and D^{-2} is a diagonal matrix. Hence, (4.2) is proved.

Provided that the condition at infinity is satisfied for $B(u_1, \ldots, u_n) = u_n - H(u_1, \ldots, u_{n-1})$ and our vectors have the form: $a_1 = e_1, a_2 = e_2, \ldots, a_{n-1} = e_{n-1}$, some $a_n \in \mathbb{R}^{n-1}$, we reduced the Bellman equation of the second sort (1.4) to a following nonlinear partial differential ineaquality on H:

$$\sum_{i,j=1}^{n-1} \frac{H_{ij}}{H_i H_j} a_{ni} a_{nj} c_{ij} \geq 0\,. \tag{4.3}$$

Corollary 4.1. *Given a vector $a_n = \{a_{nj}\}_{j=1}^k$, any function H for which there exists $k \times k$ matrix $C \geq 0$ such that $\langle Ca_n, a_n\rangle, c_{jj} > 0$, and simultaneously the PD inequality* (4.3) *holds, gives rise to an "isoperimetric inequality"* (3.4)*:*

$$\textit{If for all } x \in \mathbb{R}^k\ u_1(a_n \cdot x) - H(u_2(x_2), \ldots, u_n(x_n)) \geq 0 \textit{ then}$$

$$\textit{for all } x \in \mathbb{R}^k,\ t > 0,\ (P_t^C u_1)(a_n \cdot x) - H(P_t^C u_2(x_2), \ldots, P_t^C u_n(x_n)) \geq 0\,. \tag{4.4}$$

(again, one should check the condition at infinity that appears from (3.3)*).*

5. Further reductions in Bellman equation of the second type: PDE that rules the Prekopa–Leindler and Ehrhard inequality

Functional version of Prekopa–Leindler and Ehrhard's inequality in arbitrary dimension can be formulated as follows: if

$$\Phi^{-1}(h(\sum^{\ell} b_j x_j)) \geq \sum^{\ell} b_j \Phi^{-1}(f_j(x_j)) \quad \text{for all} \quad x_j \in \mathbb{R}^m,$$

then

$$\Phi^{-1}\left(\int_{\mathbb{R}^m} h d\gamma_m\right) \geq \sum^{\ell} b_j \Phi^{-1}\left(\int_{\mathbb{R}^m} f_j d\gamma_m\right).$$

In case of Ehrhard's inequality we require that $b_j > 0$, $\sum b_j \geq 1$, $b_j - \sum_{i\neq j} b_i \leq 1$, $h, f_j : \mathbb{R}^m \to [0,1]$ and $\Phi(x) = \int_{-\infty}^{x} d\gamma_1$, and in case of Prekopa–Leindler's inequality requirements are different: $b_j > 0$, $\sum b_j = 1$, $h, f_j : \mathbb{R}^m \to \mathbb{R}^+$ and $\Phi(x) = e^x$.

We will prove these inequalities by using second type of Bellman PDE when $m = 1$. Arbitrary dimension follows easily by iterating one dimensional case m times.

Let $k = n - 1$. Take any vector $a_n = b = (b_1, \ldots, b_k) \in \mathbb{R}^k$ such that $b_j > 0$. Take $H(x_1, \ldots, x_k) = \Phi(b_1\Phi^{-1}(x_1) + \ldots + b_k\Phi^{-1}(x_k))$ where $\Phi(x) = \int_{-\infty}^{x} \varphi(x)dx$, $\varphi > 0$, $\varphi \in C^1$ and $\Phi(x)$ is finite for any $x \in \mathbb{R}$. Let $k \times k$ matrix $C = \{c_{ij}\} \geq 0$. Clearly $\partial_j H > 0$ for all j.

Direct computations give

$$\left(\sum_{i,j=1}^{k} \frac{H_{ij}}{H_i H_j} b_i b_j c_{ij}\right) = \frac{\varphi'(\sum b_j y_j)}{\varphi(\sum b_j y_j)}\langle Cb, b\rangle - \sum_j \frac{\varphi'(y_j)}{\varphi(y_j)} b_j c_{jj}. \tag{5.1}$$

Here $y_j = \Phi(x_j)$ for all $j = 1, \ldots, n-1$.

Let us first require that $\langle Cb, b\rangle = 1$ and $c_{jj} = 1$ for all $j = 1, \ldots, k$. (Of course the existence of such $C \geq 0$ should depend on vector b.)

In order to apply Corollary 4.1 we need to have the following conditions:

A1. There exists $C \geq 0$ such that $\langle Cb, b\rangle = 1$ and $c_{jj} = 1$ for all j.

A2. Logarithmic derivative of φ satisfies the following "concavity condition":

$$(\log\varphi)'(\Sigma b_j y_j) \geq \Sigma b_j (\log\varphi)'(y_j). \tag{5.2}$$

A3. Condition at infinity (3.3) is satisfied i.e.,

$$\liminf_{|x|\to\infty} \inf_{u_j\geq 0} \frac{u_n}{1+|b\cdot x|} - \Phi\left(b_1\Phi^{-1}\left(\frac{u_1}{1+|x_1|}\right) + \ldots + b_k\Phi^{-1}\left(\frac{u_k}{1+|x_k|}\right)\right) \geq 0.$$

Then under the assumptions A1-A3, Corollary 4.1 implies: if for compactly supported functions $u_1, \ldots, u_n$ we have

$$u_n(b\cdot x) \geq \Phi(b_1\Phi^{-1}(u_1(x_1)) + \ldots + b_k\Phi^{-1}(u_k(x_k))) \quad \text{for all} \quad x \in \mathbb{R}^k,$$

then

$$\int u_n d\gamma \geq \Phi\left[b_1\Phi^{-1}\left(\int u_1 d\gamma\right) + \ldots + b_k\Phi^{-1}\left(\int u_k d\gamma\right)\right].$$

We are going to study each condition of $A1 - A3$ separately.

Condition A1. Since $C = V^*V$ for some $V = (v_1, \ldots, v_k)$ where v_j are columns of V we see that condition $c_{jj} = \langle Ce_j, e_j\rangle = 1$ implies that v_j's are unit vectors. Condition $\langle Cb, b\rangle = 1$ implies that $|\sum_{j=1}^{k} v_j b_j| = 1$. The last one gives necessary conditions $\sum_j b_j \geq 1$. Note also

that triangle inequality together with $|\sum_{j=1}^{k} v_j b_j| = 1$ implies that $1 \geq b_j - \sum_{i\neq j} b_i$. Thus we obtain two necessary conditions

$$1 \leq \sum_{j=1}^{k} b_j \quad \text{and} \quad 1 \geq b_j - \sum_{i\neq j} b_i \quad \text{for all} \quad j = 1, \ldots, k. \tag{5.3}$$

It turns out that these two conditions are also sufficient for the existence of matrix C (see Lemma 3 in [4]).

Condition A2. This condition will be just assumption on the function φ. Note that the particular function $\varphi(x) = \lambda e^{-rx^2}$ always gives us equality in (5.2). In particular, this choice will give us Ehrhard's inequality, namely the choice $\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$.

Another interesting choice is $\varphi(x) = e^x$ for which (5.2) becomes $\sum b_j \leq 1$. This together with (5.3) gives $\sum b_j = 1$. This will give us Prekopa–Leindler's inequality. Note that in this case by our choice of ϕ we have $H(x_1, \ldots, x_k) = x_1^{b_1} \cdots x_k^{b_k}$, where $b_j > 0$ and $\sum_j b_j = 1$.

Condition A3. Here we follow the same reasonings as in Example 1. Note that if $x \to \infty$ then by compactness argument, we can choose subsequence and we can assume that one of the coordinates $x_s \to \infty$. This implies that $\Phi^{-1}\left(\frac{u_s}{1+|x_s|}\right) \approx -\infty$. Thus condition A3 would be satisfied provided that none of the $\Phi^{-1}\left(\frac{u_j}{1+|x_j|}\right) \approx \infty$, $j \neq s$. For this purpose let us require that u_i's are separated from infinity in the sense of Φ, i.e., $\Phi^{-1}(|u_j|_\infty) < \infty$.

It is interesting to mention that we can get rid off Condition A1, because based on Corollary 4.1 we do not have to require $\langle Ca_j, a_j\rangle = 1$ for all j (this was necessary for the applications). In general the above considerations lead us to the following corollary.

Let $k \geq 2$. $b = (b_1, \ldots, b_k)$ and $b_j > 0$. For $\varphi > 0$ such that $\varphi \in C^1$ set $\Phi(x) = \int_{-\infty}^{x} \varphi$. Assume $\Phi(x)$ is locally finite. Let u_j be smooth functions with compact support such that $\Phi^{-1}(|u_j|_\infty) < \infty$.

Corollary 5.1. *Let $v_j \in \mathbb{R}^k$ be such that $v_j \neq 0$ and $\sum b_j v_j \neq 0$. If $\Phi(x) = \int_{-\infty}^{x} \varphi$, and*

$$|\sum b_j v_j|^2 (\log \varphi)'(\sum b_j y_j) \geq \sum b_j |v_j|^2 (\log \varphi)'(y_j) \quad \textit{for all} \quad y_j,$$

then the inequality

$$\Phi^{-1}\left(u_0(\sum b_j x_j)\right) \geq \sum b_j \Phi^{-1}(u_j(x_j)) \quad \textit{for all} \quad x \in \mathbb{R}^k,$$

implies

$$\Phi^{-1}\left(\int_{\mathbb{R}} u_0(|\sum b_j v_j| y) d\gamma(y)\right) \geq \sum b_j \Phi^{-1}\left(\int_{\mathbb{R}} u_j(|v_j| y) d\gamma(y)\right).$$

Proof. Corollary immediately follows by taking $C = V^T V$ where $V = (v_1, \ldots, v_k)$ and noticing that $\langle Ce_j, e_j\rangle = |v_j|^2$ and $\langle Cb, b\rangle = |\sum b_j v_j|^2$. □

Corollary 5.2. *Let Ω be a bounded rectangular domain in $\mathbb{R}^2$ such that $0 \in \mathrm{Cl}(\Omega)$. Let $\alpha, \beta \in \mathbb{R}$ be such that $|\alpha| + |\beta| \geq$ and $1 \geq ||\alpha| - |\beta||$. Let a smooth function $H(x, y) : \Omega \to \mathbb{R}$ be such that $\frac{\partial H}{\partial x}, \frac{\partial H}{\partial y} \neq 0$ and*

$$(1 - \alpha^2 - \beta^2)\frac{\partial H}{\partial x}\frac{\partial H}{\partial y}\frac{\partial^2 H}{\partial x \partial y} + \alpha^2 \left(\frac{\partial H}{\partial y}\right)^2 \frac{\partial^2 H}{\partial x^2} + \beta^2 \left(\frac{\partial H}{\partial x}\right)^2 \frac{\partial^2 H}{\partial y^2} \geq 0. \tag{5.4}$$

Then for smooth bounded function u_3 and compactly supported functions u_1, u_2 such that $(u_1, u_2) : \mathbb{R}^k \to \Omega$, the inequality holds

$$\int_{\mathbb{R}^k} u_3 d\gamma_k \geq H\left(\int_{\mathbb{R}^k} u_1 d\gamma_k, \int_{\mathbb{R}^k} u_2 d\gamma_k\right)$$

whenever $u_3(\alpha x + \beta y) \geq H(u_1(x), u_2(y))$ *for all* $x, y \in \mathbb{R}^k$.

Proof. The corollary is immediate consequence of Corollary 4.1. Indeed, take $n = 3$ and $k = 1$. Take $a_3 = (\alpha, \beta)$. It is clear that we should choose $c_{11} = c_{22} = 1$ and $c_{12} = \frac{1-\alpha^2-\beta^2}{4\alpha^2\beta^2}$. In this case condition $C \geq 0$ is the same as $|\alpha| + |\beta| \geq$ and $1 \geq ||\alpha| - |\beta||$. Inequality (5.4) is the same as (4.3).

Now we left to check condition at infinity 3.3. Let $|(x, y)| \to \infty$. Suppose that both $|x|, |y| \to \infty$. Since u_1 and u_2 are compactly supported this means that we need to ensure that the following inequality holds

$$u_3(z, t) \geq H(0, 0)$$

for all $z \in \mathbb{R}^k$. This follows from the pointwise inequality: since $u_3(\alpha x+\beta y) \geq H(u_1(x), u_2(y))$ then taking x, y sufficiently large we can make $\alpha x + \beta y$ to be any point $z \in \mathbb{R}^k$. Then from the pointwise inequality $u_3(y) \geq H(0, 0)$ for all $y \in \mathbb{R}^k$ we obtain integral inequality after integrating it with respect to probability measure $p_t(z, y)dy$ of the heat semigroup P_t.

Now consider the case when $|y| \to \infty$ and $|x|$ is bounded. In this case we need to show that

$$\liminf_{|y|\to\infty} u_3(y, t) \geq H(u_1(x, t), 0).$$

Notice that pointwise inequality implies $\liminf_{|y|\to\infty} u_3(y) \geq H(u_1(x), 0)$. Since $u_1(x, t) = u_1(x^*)$ for some x^* and $\liminf_{|y|\to\infty} u_3(y, t) \geq \liminf_{|y|\to\infty} u_3(y)$ we obtain the desired result.

In order to obtain the corollary for the general rank case i.e., for arbitrary $k > 1$ we can iterate the inequality as we did before in case of Ehrhard's inequality (or one can see Section 6). □

5.1. **Solving particular case of second type PDE.** In this section we will partially solve PDE (4.3) in the case $n = 3$. assume that $a_3 = (a, b) \in \mathbb{R}^2$. Let us require that for $C \geq 0$ we have $c_{jj} = 1$ and $\langle Ca_3, a_3\rangle = 1$. This can happen if and only if $|a| + |b| \geq 1$ and $||a| - |b|| \leq 1$. In other words, this can be written as one condition:

$$1 \geq \frac{(1 - a^2 - b^2)^2}{4a^2b^2}.$$

Condition $\langle Ca_3, a_3\rangle = 1$ implies that $c_{12} = \frac{1-a^2-b^2}{2ab}$. Therefore (4.3) takes the following form

$$H_1H_2H_{12}(1 - a^2 - b^2) + a^2H_2^2H_{11} + b^2H_1^2H_{22} = 0, \tag{5.5}$$

where $H = H(x, y)$. Let us show that the equation (5.5) can be reduced to second order linear differential equation with constant coefficients.

In particular, we will see that if $|c_{12}| = 1$ then the equation becomes parabolic equation and it reduces to heat equation, and if $|c_{12}| < 1$ then the equation becomes elliptic equation which reduces to Laplacian eigenfunctions.

Let $H_1 = p, H_2 = q$ (therefore $p_y = q_x$), then (5.5) becomes

$$pqp_y(1 - a^2 - b^2) + a^2q^2p_x + b^2p^2q_y = 0.$$

Assuming that the map $(x,y) \to (p(x,y), q(x,y))$ is locally invertible we obtain (exactly as we did in Section 2.3)

$$-pqx_q(1-a^2-b^2) + a^2q^2y_q + b^2p^2x_p = 0, \tag{5.6}$$

and $x_q = y_p$.

We differentiate (5.6) with respect to p:

$$-qx_q(1-a^2-b^2) - pqx_{pq}(1-a^2-b^2) + a^2q^2x_{qq} + 2b^2px_p + b^2p^2x_{pp} = 0.$$

Let $x(p,q) = B(\ln p, \ln q)$ then $p^2x_{pp} = B_{uu} - B_u$ and $q^2x_{qq} = B_{vv} - B_v$. Then

$$-B_v(1-a^2-b^2) - B_{uv}(1-a^2-b^2) + a^2(B_{vv} - B_v) + 2b^2B_u + b^2(B_{uu} - B_u) = 0.$$

Hence

$$b^2B_{uu} + B_{uv}(1-a^2-b^2) + a^2B_{vv} + B_ub^2 + B_v(b^2-1) = 0. \tag{5.7}$$

Thus we obtained second order linear PDE with constant coefficients. All we know about the numbers a, b is that

$$1 \geq \frac{(1-a^2-b^2)^2}{4a^2b^2}$$

So if $4a^2b^2 = (1-a^2-b^2)^2$ (which is the same as $|c_{12}| = 1$) then the above equation corresponds to the parabolic equation, and if $4a^2b^2 > (1-a^2-b^2)^2$ then the above equation becomes elliptic equation.

Parabolic equation: heat equation. Assume that $4a^2b^2 = (1-a^2-b^2)^2$, i.e., $b = 1-a$. Then our PDE becomes

$$(1-a)^2B_{uu} + 2a(1-a)B_{uv} + a^2B_{vv} + (1-a)^2B_u + a(a-2)B_v = 0.$$

Since this corresponds to parabolic equation we can not make coefficient in front of B_{uu} and B_{vv} zero simultaneously. So we make the following change of variables $B(u,v) = M(\frac{a}{1-a}u - v, u)$. Then

$$M_{22} + M_2 + \frac{a(3-2a)}{(1-a)^2}M_1 = 0.$$

The following technical lemma describes solutions of this PDE.

Lemma 2. If

$$M_{22} + c_1M_2 + c_2M_1 = 0$$

and $c_2 \neq 0$ then $M(x,y) = e^{-\frac{c_1y}{2} + \frac{c_1^2x}{4c_2}}W(\frac{-x}{c_2}, y)$ where W satisfies the heat equation $W_{22} = W_1$.

Elliptic equation: Laplacian eigenfunctions. In order to get rid off mixed derivatives we make change of variables as follows

$$B(u,v) = M\left(u\frac{(1-a^2-b^2)}{2b^2} - v, u\frac{\sqrt{4a^2b^2 - (1-a^2-b^2)^2}}{2b^2}\right).$$

Then the equation (5.7) becomes

$$\Delta M + M_2\frac{2b^2}{\sqrt{4a^2b^2 - (1-a^2-b^2)^2}} + M_1\frac{2b^2(3-3b^2-a^2)}{4a^2b^2 - (1-a^2-b^2)^2} = 0.$$

The following technical lemma reduces the question to Laplacian eigenfunction problem:

Lemma 3. If

$$M_{11} + M_{22} + c_1 M_1 + c_2 M_2 = 0,$$

then $M(x,y) = e^{-\frac{c_1 x}{2} - \frac{c_2 y}{2}} W(x,y)$ where W is eigenfunction of the Laplacian, i.e.,

$$\Delta W = \left(\frac{c_1^2 + c_2^2}{4}\right) W.$$

6. General rank case

6.1. **Special initial data.** So far we were considering special initial datas of the form $f(x) = F(a \cdot x)$ where $x, a \in \mathbb{R}^k$. It is natural to consider the following initial datas as well $f(x) = F(xA)$ where A is $k \times s$ matrix and $x \in \mathbb{R}^k$, and $F : \mathbb{R}^s \to \mathbb{R}$. Note that we are writing xA instead of more usual notations Ax only because we want to keep the same notations as above $a \cdot x = xa$ where a was column and x was a row.

For these initial datas absolutely nothing changes except we will work with larger matrices. Let us briefly formulate all results and leave the details. For a symmetric matrix $Q = \{q_{ij}\}$ we set $L_Q = \sum q_{ij} \frac{\partial^2}{\partial x_i \partial x_j}$. Corresponding semigroup will be denoted by P_t^Q. Further everywhere C is symmetric $k \times k$ matrix. Analog of $1D$ heat flow is (see Section 1.1)

$$\frac{\partial}{\partial t} U(y,t) = L_{A^* C A} U(y,t), \quad y \in \mathbb{R}^s, \quad U(y,0) = F(y).$$

Then

$$P_t^C f(x) = U(xA, t) = \int_{\mathbb{R}^s} F(xA + (2tA^* CA)^{1/2} y) d\gamma_s(y).$$

So in order the expressions to be justified we only need to require $A^* CA > 0$ but we do not need $C > 0$. Note that

$$(L_C - \partial_t) U(xA, t) = (L_{A^* CA} - \partial_t) U(y,t)|_{y = xA} = 0.$$

Therefore further we will be using sometimes the notation $P_t^C f(x)$ even though C is not necessarily positive however we will assume that $A^* CA > 0$. Note that

$$\nabla P_t^C f(x) = A(\nabla_y U(y,t)|_{y=xA})^T.$$

6.2. **First type of Bellman PDE for the general rank case.** Let $A_1, \ldots, A_n$ be matrices such tat A_j is $k \times k_j$ size and let $A = (A_1, \ldots, A_n)$ be $k \times (k_1 + \ldots + k_n)$ size. Let $B : \Omega \subset \mathbb{R}^k \to \mathbb{R}$ be smooth function on some rectangular domain Ω. Take any $k \times k$ symmetric matrix $C > 0$. Let $u_j : \mathbb{R}^{k_j} \to \mathbb{R}$ be smooth compactly supported functions, and let $\vec{u}(x) = (u_1(xA_1), \ldots, u_n(xA_n)) : \mathbb{R}^k \to \Omega$.

Theorem 3. The following conditions are equivalent:

(i) $A^* CA \bullet \operatorname{Hess} B \leq 0$ on Ω.
(ii) $(P_t^C B(\vec{u}))(x) \leq B((P_t^C \vec{u})(x))$ for all $t \geq 0$, $x \in \mathbb{R}^k$ and u_j.
(iii) $(P_t^C B(\vec{u}))(x) \leq B((P_t^C \vec{u})(x))$ for $t = 1/2$, $x = 0$ and for all u_j.

Here $A^* CA \bullet \operatorname{Hess} B$ denotes $(\sum k_j) \times (\sum k_j)$ matrix $\{A_i^* CA_j \partial_{ij} B\}_{i,j=1}^n$ i.e., $A^* CA \bullet \operatorname{Hess} B$ is constructed by the bloks $A_i^* CA_j \partial_{ij} B$. Note that if C and $A_j^* A_j$ are identity matrices then condition (iii) of Theorem 3 takes the form

$$\int_{\mathbb{R}^k} B(u_1(xA_1), \ldots, u_n(xA_n)) d\gamma_k(x) \leq B\left(\int_{\mathbb{R}^{k_1}} u_1(x) d\gamma_{k_1}(x), \ldots, \int_{\mathbb{R}^{k_n}} u_n(x) d\gamma_{k_n}(x)\right).$$

6.3. **Second type of Bellman PDE for the general rank case.** We use the same notations as in the previous section except instead of $C > 0$ we only assume that $C \geq 0$ and $A_j^* C A_j > 0$. Let $T = (B_1 A_1, \ldots, B_n A_n)$ be $k \times (k_1 + \ldots + k_n)$ matrix, where $B_j = \partial_j B$.

Theorem 4. Assume $P_{\ker T}(A^* C A \bullet \operatorname{Hess} B) P_{\ker T} \leq 0$. Then

$$\begin{aligned} \text{if} \quad & B((P_t^C \vec{u})(x)) \geq 0 \quad \text{for} \quad t = 0 \quad \text{and} \quad \forall x \in \mathbb{R}^k, \\ \text{then} \quad & B((P_t^C \vec{u})(x)) \geq 0 \quad \text{for} \quad t \geq 0 \quad \text{and} \quad \forall x \in \mathbb{R}^k, \end{aligned}$$

provided that condition at infinity holds:

$$\liminf_{|x| \to \infty} \inf_{u_j} B\left(\frac{u_1}{1 + |xA_1|}, \ldots, \frac{u_n}{1 + |xA_n|}\right) \geq 0.$$

6.4. **Applications tensorizes.** We remind that Borell's Gaussian noise stability (see Section 2) and hypercontractivity of Ornstein–Uhlenbeck (see Section 2.1) were consequences of inequality (2.3) which in turn is equivalent to PDE (2.2). Let us show that the same function implies these results in arbitrary dimension. Namely it is enough to show that if B satisfies (2.2) then

$$\int_{\mathbb{R}^{2n}} B(f(x), g(px + \sqrt{1-p^2}y)) d\gamma_2(x, y) \leq B\left(\int_{\mathbb{R}^n} f(x) d\gamma_n(x), \int_{\mathbb{R}^n} g(x) d\gamma_n(x)\right).$$

Indeed, we will apply Theorem 3 for $A_1 = (I_{n \times n}, 0_{n \times n})^T$, $A_2 = (pI_{n \times n}, \sqrt{1-p^2} I_{n \times n})^T$ and $C = I_{n \times n}$. Here $I_{n \times n}$ is $n \times n$ identity matrix and $0_{n \times n}$ is $n \times n$ zero matrix. Then

$$A_1^* A_1 = A_2^* A_2 = I_{n \times n} \quad \text{and} \quad A_1^* A_2 = A_2^* A_1 = pI_{n \times n}.$$

Therefore condition $A^* C A \bullet \operatorname{Hess} B \leq 0$ becomes

$$\begin{pmatrix} B_{11} & pB_{12} \\ pB_{12} & B_{22} \end{pmatrix} \otimes I_n \leq 0,$$

and this is equivalent to (2.2)

7. Short review of some classical isoperimetric inequalities

Brunn–Minkowski and isoperimetric inequalities. Let A and B be nonempty compact subsets of $\mathbb{R}^n$.

Theorem. *The following sharp Brunn–Minkowski inequality holds*

$$|A + B|^{1/n} \geq |A|^{1/n} + |B|^{1/n},$$

where $n \geq 1$ and $|A|$ denotes Lebesgue measure of the set A.

The Brunn-Minkowski inequality is a consequence of its multiplicative version:

Theorem. *Let $\lambda \in (0,1)$. Then for any compact measurable sets $U, V \subset \mathbb{R}^n$ we have*

$$|\lambda U + (1-\lambda)V| \geq |U|^{\lambda} |V|^{1-\lambda}. \tag{7.1}$$

Indeed, if one sets $U\lambda = A$ and $(1-\lambda)V = B$ then inequality (7.1) takes the form

$$|A+B| \geq \frac{|A|^{\lambda}|B|^{1-\lambda}}{\lambda^{\lambda n}(1-\lambda)^{(1-\lambda)n}}. \tag{7.2}$$

By maximizing the right hand side of (7.2) over $\lambda \in (0,1)$ we obtain the Brunn–Minkowski inequality.

Brunn–Minkowski inequality implies the classical isoperimetric inequality:

Theorem. *Among all simple closed surfaces with given surface area, the sphere encloses a region of maximal volume. In other words*

$$|\partial A| \geq n|A|^{1-\frac{1}{n}}|B(0,1)|^{\frac{1}{n}}.$$

Where $|\partial A|$ means surface area of the boundary of the body A. $|A|$ denotes volume of the body and $B(0,1)$ denotes the ball of radius 1 at center 0.

Indeed, let us sketch the proof: Since $|A+B(0,\varepsilon)| = |A| + \varepsilon|\partial A| + O(\varepsilon^2)$, we have

$$|\partial A| = \lim_{\varepsilon\to 0}\frac{|A+B(0,\varepsilon)|-|A|}{\varepsilon} \geq \lim_{\varepsilon\to 0}\frac{(|A|^{1/n}+|B(0,\varepsilon)|^{1/n})^n-|A|}{\varepsilon} = n|A|^{1-\frac{1}{n}}|B(0,1)|^{\frac{1}{n}}.$$

For the possible references we refer the reader to [1, 5, 29]

Sobolev inequality. It is known that the classical isoperimetric inequality is equivalent to its functional version, to Sobolev inequality on $\mathbb{R}^n$ with optimal constant

$$\int_{\mathbb{R}^n}|\nabla f| \geq n|B(0,1)|^{\frac{1}{n}}\left(\int_{\mathbb{R}^n}|f|^{\frac{n}{n-1}}\right)^{1-\frac{1}{n}}. \tag{7.3}$$

Indeed, testing (7.3) over characteristic functions $f(x) = \mathbf{1}_A(x)$ we obtain implication in one direction. Opposite direction follows from Coarea formula: assume $f \geq 0$ is sufficiently nice compactly supported function. Then by coarea formula we have

$$\int_{\mathbb{R}^n}|\nabla f|dx = \int_0^{\infty}|\{x\,:\, f(x)=t\}|dt \geq n|B(0,1)|^{\frac{1}{n}}\int_0^{\infty}|\{x\,:\, f(x)\geq t\}|^{1-\frac{1}{n}}dt.$$

It is left to show that

$$\left(\int_0^{\infty}|\{x\,:\, f(x)\geq t\}|^{\frac{n-1}{n}}dt\right)^{\frac{n}{n-1}} \geq \frac{n}{n-1}\int_0^{\infty}|\{x\,:\, f(x)\geq t\}|t^{\frac{1}{n-1}}dt$$

This follows from the following observation

$$F\left(\int_0^{\infty}\varphi\right) = \int_0^{\infty}\frac{d}{dt}F\left(\int_0^t\varphi\right)dt = \int_0^{\infty}F'\left(\int_0^t\varphi\right)\varphi dt \geq \int_0^{\infty}F'(t\varphi(t))\varphi(t)dt,$$

where φ is decreasing and F' is increasing ($F(t) = t^{\frac{n}{n-1}}$, $\varphi(t) = |\{x\,:\, f(x)\geq t\}|^{\frac{n-1}{n}}$). So the claim follows.

Prekopa–Leindler inequality. Multiplicative Brunn–Minkowski inequality follows from its functional version, so called Prekopa–Leindler inequality.

Theorem. *Let h, f, g be positive measurable functions and $\lambda \in (0,1)$. If*

$$h(\lambda x + (1-\lambda)y) \geq f(x)^{\lambda} g(y)^{1-\lambda} \tag{7.4}$$

Then

$$\int_{\mathbb{R}^n} h \geq \left(\int_{\mathbb{R}^n} f\right)^{\lambda} \left(\int_{\mathbb{R}^n} g\right)^{1-\lambda}.$$

If one takes $h(x) = \mathbf{1}_{\lambda \mathbf{U}+(\mathbf{1}-\lambda)\mathbf{V}}(\mathbf{x}), \mathbf{f}(\mathbf{x}) = \mathbf{1}_{\mathbf{U}}(\mathbf{x})$ and $g(x) = \mathbf{1}_{\mathbf{V}}(\mathbf{x})$ then clearly the assumption (7.4) is satisfied and one obtains multiplicative version of Brunn–Minkowski inequality.

Straightforward generalization of Prekopa–Leindler inequality takes the following form:

Theorem. *Let $f_j : \mathbb{R}^n \to R_+$ be integrable functions, and let $\sum_{j=1}^m \lambda_j = 1$, $0 < \lambda_j < 1$. If*

$$h\left(\sum_{j=1}^m \lambda_j x_j\right) \geq \prod_{j=1}^m f(x_j)^{\lambda_j},$$

then

$$\int_{\mathbb{R}^n} h \geq \prod_{j=1}^m \left(\int_{\mathbb{R}^n} f_j\right)^{\lambda_j}.$$

The above inequality can be treated as reverse to Hölder's inequality:

$$\int_{\mathbb{R}^n} \sup\left\{\prod_{j=1}^m f(x_j)^{\lambda_j} \,:\, \sum x_j \lambda_j = z\right\} dz \geq \prod_{j=1}^m \left(\int_{\mathbb{R}^n} f_j\right)^{\lambda_j} \geq \prod_{j=1}^m \int_{\mathbb{R}^n} f_j(x_j)^{\lambda_j}.$$

where integral in the left hand side is understood as upper Lebesgue integral.

Note that we proved Prekopa–Leindler inequality in Section 5 when $\Phi(x) = e^x$ (see discussions given after the explanation of *Condition A2*). Basically the reason inequality holds is because the function $H(x_1, \ldots, x_m) = \prod_{j=1}^m x_j^{\lambda_j}$ satisfies partial differential inequality (4.3) for appropriate choice of C and $a_n = (\lambda_1, \ldots, \lambda_m)$.

One of the other applications of Prekopa–Leindler inequality in probability is that:

Corollary. *If $F(x,y) : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^+$ is log-concave distribution i.e.,*

$$F(\lambda u + (1-\lambda)v) \geq F(u)^{1-\lambda} F(v)^{\lambda} \quad \textit{for all} \quad u, v \in \mathbb{R}^{n+m},$$

then $H(x) = \int_{\mathbb{R}^m} F(x,y)dy$ is log-concave distribution.

The corollary immediately follows from application of Prekopa–Leindler inequality to the functions $F(x, \lambda y_1 + (1-\lambda)y_2), F(x, y_1)$ and $F(x, y_2)$.

Borell–Brascamp–Lieb inequality. We also mention Borell–Brascamp–Lieb inequality since it generalizes Prekopa–Leindler inequality

Theorem. *Let h, f, g be nonnegative functions, $0 < \lambda < 1$ and $-\frac{1}{n} \leq p \leq \infty$. Suppose*

$$h(\lambda x + (1-\lambda)y) \geq M_p(f(x), g(y), \lambda),$$

where

$$M_p(a,b,\lambda) := (\lambda a^p + (1-\lambda)b^p)^{1/p}, \quad M_0 := (a,b,\lambda) = a^\lambda b^{1-\lambda}.$$

Then

$$\int_{\mathbb{R}^n} h \geq M_{\frac{p}{np+1}}\left(\int_{\mathbb{R}^n} f, \int_{\mathbb{R}^n} g, \lambda\right). \tag{7.5}$$

Notice that $H(x,y) = M_p(x,y,\lambda)$ satisfies partial differential inequality (5.4) for $p \geq 1$ (here $(\alpha,\beta) = (\lambda, 1-\lambda)$). Indeed,

$$(1-\alpha^2-\beta^2)\frac{\partial H}{\partial x}\frac{\partial H}{\partial y}\frac{\partial^2 H}{\partial x \partial y} + \alpha^2\left(\frac{\partial H}{\partial y}\right)^2\frac{\partial^2 H}{\partial x^2} + \beta^2\left(\frac{\partial H}{\partial x}\right)^2\frac{\partial^2 H}{\partial y^2} =$$
$$(p-1)\frac{\lambda(1-\lambda)(x^p-y^p)^2}{(xy)^p H(x,y)} \geq 0$$

Thus by Corollary 5.2 we obtain

$$\int_{\mathbb{R}^n} h \geq M_p\left(\int_{\mathbb{R}^n} f, \int_{\mathbb{R}^n} g, \lambda\right). \tag{7.6}$$

Also notice that $M_p(x,y,\lambda) \geq M_{\frac{p}{np+1}}(x,y,\lambda)$ for $x, y \geq 0$ and $-\frac{1}{n} < p < \infty$ (this is a direct computation: by homogeneity we can assume that $x = 1$, and the rest follows by showing that the derivative of the function $f(y) = (\lambda + (1-\lambda)y^p)^{1/p} - (\lambda + (1-\lambda)y^{\frac{p}{np+1}})^{\frac{np+1}{p}}$ has only one root $y = 1$).

Thus inequality (7.6) is better than (7.5), and hence it implies Borell–Brascamp–Lieb inequality in case $p \geq 1$.

In the case $-\frac{1}{n} \leq p \leq 1$ we do not know how to derive Borell–Brascamp–Lieb inequality by using Corollary 5.2. The reason is because the inequality (5.4) does not hold if $p < 1$.

Ehrhard's inequality. The condition of Prekopa–Leindler type appears in Ehrhard's inequality (see [10, 21]):

Theorem. *Let $d\gamma(x) = \frac{e^{-|x|^2/2}}{(2\pi)^{n/2}}dx$ be the Gaussian measure. And let $\Phi(x) = \int_{-\infty}^x d\gamma$. Then for any measurable compact sets $A, B \subset \mathbb{R}^n$ and any numbers $\lambda, \mu \geq 0$, such that $\lambda + \mu \geq 1$ and $|\lambda - \mu| \leq 1$ we have*

$$\Phi^{-1}(|\lambda A + \mu B|_\gamma) \geq \lambda\Phi^{-1}(|A|_\gamma) + \mu\Phi^{-1}(|B|_\gamma), \tag{7.7}$$

where $|A|_\gamma$ denotes Gaussian measure of A i.e., $|A|_\gamma = \int_A d\gamma$.

The inequality initially was stated for convex sets A and B. Later it was improved in the sense that only one of them has to be convex and it was conjectured that the inequality is true in general for any measurable sets, and the conjecture was proved recently. Ehrhard's inequality is consequence of its functional version:

Theorem. *Let $h, f, g : \mathbb{R}^n \to [0,1]$ be functions such that*

$$\Phi^{-1}(h(\lambda x + \mu y)) \geq \lambda\Phi^{-1}(f(x)) + \mu\Phi^{-1}(g(y)), \quad \textit{for all} \quad x, y \in \mathbb{R}^n,$$

where $\lambda, \mu \geq 0$, $\lambda + \mu \geq 1$ and $|\lambda - \mu| \leq 1$ then

$$\Phi^{-1}\left(\int_{\mathbb{R}^n} h d\gamma\right) \geq \lambda\Phi^{-1}\left(\int_{\mathbb{R}^n} f d\gamma\right) + \mu\Phi^{-1}\left(\int_{\mathbb{R}^n} g d\gamma\right).$$

Note that we proved Ehrhard's inequality in Section 5, and the reason the inequality holds was because the function

$$H(x,y) = \Phi\left(\alpha\Phi^{-1}(x) + \beta\Phi^{-1}(y)\right)$$

(where $\Phi(x) = \int_{-\infty}^{x} d\gamma(x)$) satisfies partial differential inequality (5.4). Ehrhard's inequality

implies Gaussian isoperimetry, which in turn follows from its integral version:

Corollary. *Let A be a Borel set in $\mathbb{R}^n$ and let H be an affine halfspace such that $\gamma_n(A) = \gamma_n(H) = \Phi(a)$ for some $a \in \mathbb{R}$. Then*

$$\gamma_n(A_t) \geq \gamma_n(H_t) = \Phi(a+t) \quad \forall t \geq 0. \tag{7.8}$$

where $A_t = A + B(t)$, and $B(t)$ is a ball of radius t centered at the origin.

Proof follows using Ehrhard's inequality (7.7):

$$|A_t|_\gamma = \left|\lambda[\lambda^{-1}A] + (1-\lambda)[(1-\lambda)^{-1}tB]\right|_\gamma \geq \Phi\left(\lambda\Phi^{-1}(|\lambda^{-1}A|_\gamma) + (1-\lambda)\Phi^{-1}(|(1-\lambda)^{-1}tB|_\gamma)\right).$$

If we send $\lambda \to 1^-$ then $(1-\lambda)^{-1}\Phi^{-1}(|(1-\lambda)^{-1}tB|_\gamma) \to t$. Indeed, we need to show that $\lim_{r\to\infty}\frac{1}{r}\Phi^{-1}(|B(r)|_\gamma) = 1$. This follows from the following asymptotic behavior of Gaussian distributions

$$|B(r)|_{\gamma_n} = 1 - \frac{|\sigma_n|}{(2\pi)^{n/2}}\int_r^\infty e^{-\frac{r^2}{2}} r^{n-1}dr = 1 - r^{n-2}e^{-r^2/2}\frac{|\sigma_n|}{(2\pi)^{n/2}} + o\left(r^{n-2}e^{-r^2/2}\right).$$

where $|\sigma_n|$ is the measure of the unit sphere in $\mathbb{R}^n$, and

$$\Phi(t) = 1 - \frac{1}{\sqrt{2\pi}}\int_t^\infty e^{-x^2/2}dx = 1 - \frac{1}{\sqrt{2\pi}}\frac{e^{-t^2/2}}{t} + o\left(\frac{e^{-t^2/2}}{t}\right).$$

Thus

$$\Phi^{-1}(s) = (-2\ln(1-s))^{1/2} + o\left((-\ln(1-s))^{1/2}\right) \quad \text{for} \quad r \to 1^-,$$

and hence

$$\lim_{r\to\infty}\frac{1}{r}\Phi^{-1}(|B(r)|_\gamma) = \lim_{r\to\infty}\frac{1}{r}\left[-2\ln(r^{n-2}e^{-r^2/2})\right]^{1/2} = 1.$$

So we obtain the desired result

$$|A_t|_\gamma \geq \Phi(\Phi^{-1}(|A|_\gamma) + t).$$

Infinitisimal version of (7.8) gives Gaussian isoperimetry

Corollary 7.1.

$$|\partial A|_\gamma := \lim_{t\to 0}\frac{|A_t|_\gamma - |A|_\gamma}{t} \geq \Phi'(\Phi^{-1}(|A|_\gamma)).$$

Borell's Gaussian noise "stability". Let $\gamma_n = \frac{e^{-|x|^2/2}}{(2\pi)^{n/2}}$ be a standard Gaussian measure on $\mathbb{R}^n$ and let $\Phi = \int_{-\infty}^{x} d\gamma_1$. Borell's Gaussian noise "stability" (see also [23, 24]) states that

Theorem. *If A, B are measurable subsets of $\mathbb{R}^n$. Then if $X = (X_1, \ldots, X_n), Y = (Y_1, \ldots, Y_n)$ are independent Gaussian standard random variables, and $p \in (0,1)$ then*

$$\mathbb{P}(X \in A,\ pX + \sqrt{1-p^2}\,Y \in B) \leq \mathbb{P}(X_1 \leq \Phi^{-1}(\gamma_n(A)),\ pX_1 + \sqrt{1-p^2}\,Y_1 \leq \Phi^{-1}(\gamma_n(B))).$$

The functional version of the above inequality can be stated as follows:

Theorem. *Let $p \in (0,1)$, $f, g : \mathbb{R}^n \to (0,1)$ and let*

$$B(u,v) = \mathbb{P}(X_1 \leq \Phi^{-1}(u),\ pX_1 + \sqrt{1-p^2}\,Y_2 \leq \Phi^{-1}(v)).$$

Then

$$\int_{\mathbb{R}^{2n}} B\left(f(x), g(px + \sqrt{1-p^2}\,y)\right) d\gamma d\gamma \leq B\left(\int_{\mathbb{R}^n} f d\gamma, \int_{\mathbb{R}^n} g d\gamma\right).$$

Hypercontractivity. Let

$$P_t f(x) = \int_{\mathbb{R}^n} f(e^{-t}x + \sqrt{1-e^{-2t}}\,y) d\gamma(y)$$

be Ornstein–Uhlenbeck semigroup where $t \geq 0$. The hypercontractivity for Ornstein–Uhlenbeck semigroup means that

Theorem. *Let $p, q > 1$ be such that $\frac{q-1}{p-1} \geq e^{-2t}$. Then*

$$\|P_t f\|_{L^p(d\gamma)} \leq \|f\|_{L^q(d\gamma)}.$$

For possible references we refer the reader to [14, 22, 12]. For the proofs we refer the reader to Section 2 (for the case $n = 1$) and to Section 6.2 for arbitrary $n \geq 1$.

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